ALGEBRAIC TOPOLOGY

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Preamble. This document contains some exercises in algebraic topology, category theory, and homological algebra. Most of them can be found as chapter exercises in Hatcher's book on algebraic topology. Use at your own risk.

> **Exercise 1.** *If a topological space X is contractible, then it is path-connected.*

Proof. Let F be the homotopy between the identity map and the constant map $f(x) = x_0$ for some contraction point $x_0 \in X$. For all $x \in X$, let $\gamma_x = F|_{\{x\}\times I}$. γ_x is continuous since restrictions of continuous functions are continuous. $\gamma_x(0) = F(x,0) = \mathrm{id}(x) = x$ and $\gamma_x(1) = F(x,1) = f(x) = x_0$, hence it is a well-defined path from x to x_0. Thus, for any $x_1, x_2 \in X$, there exists a path from x_1 to x_2 given by $\widehat{\gamma_{x_2}} * \gamma_{x_1}$. [1] ■

> **Exercise 2.** *If $f, f' : X \to Y$ are homotopic and $g, g' : Y \to Z$ are homotopic, then $g \circ f$ and $g' \circ f'$ are homotopic.*

Proof. Let F and G denote the homotopies between f, f' and g, g' respectively. Let $H(x,t) = G(F(x,t),t)$. H is continuous and $H(x,0) = G(F(x,0),0) = G(f(x),0) = g(f(x))$ and $H(x,1) = G(F(x,1),1) = G(f'(x),1) = g'(f'(x))$, hence $g \circ f$ and $g' \circ f'$ are homotopic. ■

> **Exercise 3.** *(a) The composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Homotopy equivalence is an equivalence relation. (b) The relation of homotopy among maps $X \to Y$ is an equivalence relation. (c) A map homotopic to a homotopy equivalence is a homotopy equivalence.*

Proof. (a) Let $f : X \to Y$ and $g : Y \to X$ be continuous functions such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. Similarly, let h, k be continuous functions such that $h \circ k \sim \mathrm{id}_Z$ and $k \circ h \sim \mathrm{id}_Y$. Composing functions we have $(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \sim g \circ \mathrm{id}_Y \circ f \sim g \circ f \sim \mathrm{id}_X$ and $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \sim h \circ \mathrm{id}_Y \circ k \sim h \circ k \sim \mathrm{id}_Z$, where we have used associativity of function composition and that homotopy is well-behaved with respect to function composition **(c)**. Since homotopy equivalence is reflective, symmetric, and transitive, it is an equivalence relation.

(b) For any map f, f is homotopic to itself by the homotopy that is f identically throughout the unit interval. If $f \sim g$ by $F(x,t)$, then $g \sim f$ by $F(x, 1-t)$. Lastly, homotopy is transitive since homotopies can be pasted together by doubling speeds and gluing. Hence, homotopy among functions is an equivalence relation.

(c) It suffices to show that if $f \sim f'$, then $f \circ g \sim f' \circ g$. But if $F(x,t)$ is the homotopy from f to f', then $F \circ g \times \mathrm{id}$ is the homotopy from $f \circ g$ to $f' \circ g$. ■

[1] A wide hat over a path denotes the inverse path. That is, $\widehat{\gamma(s)} = \gamma(1-s)$

Exercise 4. *(a) A space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is null-homotopic. (b) Similarly, X is contractible iff every map $f : Y \to X$ is null-homotopic.*

Proof. (a) $(\implies)$ Let F be a contraction of X to some contraction point x_0. Then for any map $f : X \to Y$, we have $f \circ F : X \times I \to Y$ giving the homotopy between f and the constant map $x \mapsto f(x_0)$. $(\impliedby)$ If every map $f : X \to Y$ for arbitrary Y is null-homotopic, then in particular id_X is null-homotopic.

(b) $(\implies)$ Let F be a contraction of X to some contraction point x_0. Then for any map $f : Y \to X$, $F(f(y), t)$ is the desired homotopy from f to $y \mapsto x_0$. $(\impliedby)$ If every map $f : Y \to X$ for arbitrary Y is null-homotopic, then in particular id_X is null-homotopic. ∎

Exercise 5. *(a) $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \cong 1$ and $hf \cong 1$. (b) More generally, f is a homotopy equivalence if fg and hf are homotopy equivalences.*

Proof. (a) f is a homotopy equivalence since $f(hfg) = f(hf)g \cong f(1)g \cong fg \cong 1$ and $(hfg)f = h(fg)f \cong h(1)f \cong 1$. (b) This is entirely similar to the proof of (a). ∎

Exercise 6. *For a path-connected space X, $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.*

Proof. $(\implies)$ Consider any two paths h, h' from x_0 to x_1. Consider any loop γ based at x_0. We have $\beta_h[\gamma] = [h\gamma h^{-1}]$ and $\beta_{h'}[\gamma] = [h'\gamma h'^{-1}]$. Conjugating again, we have $[h^{-1}]\beta_h[\gamma][h] = [\gamma]$ and $[h'^{-1}]\beta_{h'}[\gamma][h'] = [\gamma]$. Hence, $\beta_h[\gamma] = [hh'^{-1}]\beta_{h'}[\gamma][h'h^{-1}] = [hh'^{-1}][h'h^{-1}]\beta_{h'}[\gamma] = \beta_{h'}[\gamma]$, since $\pi_1(X)$ is abelian.

$(\impliedby)$ Let γ_1 and γ_2 be two loops based at x_0. Decompose and reparametrize γ_1 into two paths $\delta_1 = \gamma([0, \frac{1}{2}])$ and $\delta_2 = \gamma([\frac{1}{2}, 1])$. By assumption, we have that $\beta_{\delta_1}[\gamma_2] = \beta_{\delta_2^{-1}}[\gamma_2]$. Explicitly, this gives $[\delta_1\gamma_2\delta_1^{-1}] = [\delta_2^{-1}\gamma_2\delta_2]$. Multiplying on the right by δ_1 and on the left by δ_2 gives $[\delta_2\delta_1\gamma_2] = [\gamma_2\delta_2\delta_1]$. But $\delta_2\delta_1 = \gamma_1$, so $\pi_1(X)$ is abelian. ∎

Exercise 7. *For a space X, the following three conditions are equivalent: (a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point. (b) Every map $S^1 \to X$ extends to a map $D^2 \to X$. (c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.*

(d) It then follows that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [Here, âĂŸhomotopicâĂŹ means âĂŸhomotopic without regard to basepointsâĂŹ.]

Proof. $(a \implies b)$ Let $f : S^1 \to X$ be any map and let h denote a homotopy from a constant map to f. Then the extension of f is just given by the homotopy, $\widetilde{f}(\theta, r) = h(\theta, r)$, where θ, r give the usual angle-radius parametrization of the disk. For $r = 1$, we have $\widetilde{f}(\theta, 1) = h(\theta, 1) = f(\theta)$.

$(b \implies c)$ Let x_0 be any point in X. Given an equivalence class in $\pi_1(X, x_0)$, a representative γ is a map $S^1 \to X$, so it extends to a map $D^2 \to X$, but the map it extends to is exactly a based homotopy to a constant loop. Hence, every loop based at x_0 is null-homotopic, so $\pi_1(X, x_0)$ is trivial.

$(c \implies a)$ The hypothesis $\pi_1(X, x_0)$ for all $x_0 \in X$ implies that all maps $S^1 \to X$ homotopic to the trivial loop, and hence homotopic to a constant map.

(d) ($\implies$) If a space is simply connected, then $\pi_1(X) = 0$, so (c) holds, and thus (a) holds. But since X is path-connected, all maps homotopic to a constant map are homotopic. ($\impliedby$) If all maps $S^1 \to X$ are homotopic, then in particular, the constant maps are homotopic, and hence X is path-connected. Additionally, (a) holds, which implies $\pi_1(X) = 0$. ■

Exercise 8. *From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$, it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. The following is an explicit homotopy demonstrating this.*

Proof. Let $\gamma_x : I \to X \times \{y_0\}$ and $\gamma_y : I \to \{x_0\} \times Y$ be loops based at (x_0, y_0). Let

$$\delta_x(t, s) = \begin{cases} x_0 & 0 \leqslant 2t \leqslant s \\ \pi_1(\gamma_x(2t - s)) & s \leqslant 2t \leqslant 1 + s \\ x_0 & 1 + s \leqslant 2t \leqslant 2 \end{cases}$$

$$\delta_y(t, s) = \begin{cases} y_0 & 0 \leqslant 2t \leqslant 1 - s \\ \pi_2(\gamma_y(2t - s)) & 1 - s \leqslant 2t \leqslant 2 - s \\ x_0 & 2 - s \leqslant 2t \leqslant 2 \end{cases}$$

Let $H(t, s) = (\delta_x(t, s), \delta_y(t, s))$. H is continuous as δ_x and δ_y are continuous. $H(t, 0) = (\delta_x(t, 0), \delta_y(t, 0)) = \gamma_x\gamma_y$ and $H(t, 1) = \gamma_y\gamma_x$ and $H(0, s) = H(1, s) = (x_0, y_0)$, so H is a based homotopy and $[\gamma_x\gamma_y] = [\gamma_y\gamma_x]$. ■

Exercise 9. *For a covering map $p : \widetilde{X} \to X$ and a subspace $A \subset X$, let $\widetilde{A} = p^{-1}(A)$. The restriction $p : \widetilde{A} \to A$ is a covering map.*

Proof. Since p is a covering map, there exists an open cover $\{U_\alpha\}$ of X by evenly covered sets U_α. That is, for all α, $p^{-1}(U_\alpha) = \bigsqcup_{i \in I} V_\alpha^i$ for some index set I, and the restriction $p : V_\alpha^i \to U_\alpha$ for any particular i is a homeomorphism. Since the U_α cover X, $\{U_\alpha \cap A\}$ is an open cover of A. Additionally, for all α, $p^{-1}(U_\alpha \cap A) = p^{-1}(U_\alpha) \cap p^{-1}(A) = \bigsqcup_{i \in I} V_\alpha^i \cap p^{-1}(A)$. $p : V_\alpha^i \to U_\alpha$ is a homeomorphism, so the restriction $p : V_\alpha^i \cap p^{-1}(A) \to p(V_\alpha^i \cap p^{-1}(A))$ is also a homeomorphism. But $p(V_\alpha^i \cap p^{-1}(A)) = U_\alpha \cap A$ since a homeomorphism is in particular bijective, so the $U_\alpha \cap A$ are evenly covered. Hence A has an open cover by evenly covered sets, so the restriction $p : p^{-1}(A) \to A$ is a covering map. ■

Exercise 10. *Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. If $X \simeq Y$, then $\widetilde{X} \simeq \widetilde{Y}$.*

Proof. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering maps and let $f : X \to Y$ and $g : Y \to X$ be homotopy equivalences such that $fg \cong \mathrm{id}_Y$ and $gf \cong \mathrm{id}_X$. Since p is a covering map, $(fp)_*(\pi_1(\widetilde{X}))$ corresponds to the trivial subgroup, and X and Y are path-connected and locally-path connected, so by the lifting criterion, fp, and similarly gq, extend to lifts $\widetilde{fp} : \widetilde{X} \to \widetilde{Y}$ and $\widetilde{gq} : \widetilde{Y} \to \widetilde{X}$ such that $q\widetilde{fp} = fp$ and $p\widetilde{gq} = gq$.

Since $p\widetilde{gq}\widetilde{fp} = gq\widetilde{fp} = gfp$, by the lifting lemma applied to the homotopy $gf \cong 1$, there exists a homotopy $p\widetilde{gq}\widetilde{fp}$ to p. This homotopy also lifts to a homotopy from $\widetilde{gq}\widetilde{fp}$ to $\widetilde{p} : \widetilde{X} \to \widetilde{X}$., where $\widetilde{p}$ is a lift of the covering map p.

But $\widetilde{p}$ is a deck transformation, since $p\widetilde{p} = p$ and $\widetilde{p}$ is a homeomorphism.

Hence, $\widetilde{p}^{-1}\widetilde{g}\widetilde{q}\widetilde{fp} \cong \widetilde{p}^{-1}\widetilde{p} = \mathrm{id}_{\widetilde{X}}$. A similar construction gives $\mathrm{id}_{\widetilde{Y}}$, and hence by **Proposition 5**, $\widetilde{X}$ and $\widetilde{Y}$ are homotopy equivalent under $\widetilde{fp}$ and $\widetilde{gq}$. ■

Lemma 1. *Let $p : X \to Y$ be a covering map and let Y be locally path-connected. Then X is locally path-connected.*

Proof. Consider any point $x \in X$ and an open neighborhood U of x. Let V denote an evenly covered neighborhood of $p(x)$. Let W be an open neighborhood of x such that $p : W \to V$ is a homeomorphism. $U \cap W$ is then homeomorphic to $p(U \cap W)$, which contains $p(x)$. But Y is path-connected, so there exists an open neighborhood $O \subset p(U \cap W)$ that is path-connected. Then the inverse image of O under the local homeomorphism is a path-connected open neighborhood of x that is a subset of U, so X is locally path-connected. ■

> **Exercise 11.** *For a covering map $p : \widetilde{X} \to X$ with X connected, locally path-connected, and semi-locally simply-connected, (a) the components of $\widetilde{X}$ are in one-to-one correspondence with the orbits of the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$. (b) Under the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of $\widetilde{X}$ containing a given lift $\widetilde{x}_0$ of x_0 is the stabilizer of $\widetilde{x}_0$, the subgroup consisting of elements whose action on the fibers leaves $\widetilde{x}_0$ fixed.*

Proof. (*a*) By **Lemma 1**, each connected component of $\widetilde{X}$ is locally path-connected, and hence path-connected. Consider any two elements $\widetilde{x}_1$ and $\widetilde{x}_2$ of the fiber $p^{-1}(x_0)$ in the same component of $\widetilde{X}$. Since the component is path-connected, there exists a path γ from $\widetilde{x}_1$ to $\widetilde{x}_2$. But then $[p\gamma] \in \pi_1(X, x_0)$, and $([p\gamma]) \cdot (\widetilde{x}_1) = \widetilde{x}_2$, hence $\widetilde{x}_1$ and $\widetilde{x}_2$ are in the same orbit. Now conversely, suppose $\widetilde{x}_1$ and $\widetilde{x}_2$ are points in the fiber $p^{-1}(x_0)$ and are in the same orbit. Explicitly, there exists $[\gamma] \in \pi_1(X, x_0)$ such that $[\gamma] \cdot \widetilde{x}_1 = \widetilde{x}_2$. But then by the lifting lemma, γ lifts to a path from $\widetilde{x}_1$ to $\widetilde{x}_2$, so $\widetilde{x}_1$ and $\widetilde{x}_2$ must be in the same component.

(*b*) Let $\widetilde{x}_0$ be a given lift of x_0. The stabilizer of $\widetilde{x}_0$ is the set of all loops classes $[\gamma]$ such that $[\gamma] \cdot \widetilde{x}_0 = \widetilde{x}_0$. But then any such γ must lift to a loop based at $\widetilde{x}_0$. These are precisely the elements in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. Each statement is reversible, so the double containment holds. ■

> **Exercise 12.** *Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy $\widetilde{f}_t$ that is stationary on both boundary circles.*

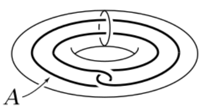
Proof. f_t is given explicitly by $f_t(\theta, s) = (\theta + 2\pi ts, s)$. Suppose for a contradiction that a homotopy $\widetilde{f}_t$ from f to the identity fixing both boundary circles existed. Then $\widetilde{f}_t$ gives a based homotopy from the trivial loop to the generator of $\pi_1(S^1)$, $(\pi \circ \widetilde{f}_0 \circ i)(s) = 0$ to $(\pi \circ \widetilde{f}_1 \circ i)(s) = 2\pi s$, which cannot exist.[2] ■

> **Exercise 13.** *Every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism ϕ_* of a map $\phi : S^1 \to S^1$.*

[2] Here i denotes the inclusion map, and π has been overloaded as the natural projection map, the ratio of the circumference of a circle to its diameter, and as the fundamental group when sub-scripted by 1.

Proof. Let $\psi : \pi_1(S) \to \pi_1(S)$ be any homomorphism. ψ is determined by the image of the generator, $\psi([\gamma_1]) = [\gamma_k]$, since $\pi_1(S) \cong \mathbb{Z}$. Let $\phi : e^{i\theta} \mapsto e^{ik\theta}$. If $\gamma_1(s) = e^{2\pi s}$, then $\phi \circ \gamma_1 = \gamma_k$, so $\phi_* = \psi$. ■

Exercise 14. *There are no retractions $r : X \to A$ in the following cases:*
(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
(c) $X = S^1 \times D^2$ and A the circle shown in the figure.



(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
(f) X the MÃűbius band and A its boundary circle.

Proof. (a) If such a retract existed, there would be a surjective homomorphism $\pi_1(\mathbb{R}^3) \cong 1 \to \pi_1(S^1) \cong \mathbb{Z}$.

(b) If such a retract existed, there would be a surjective homomorphism from $\pi_1(S^1 \times D^2) =\cong \mathbb{Z} \times \{0\} \cong \mathbb{Z} \to \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

(c) The generator of $\pi_1(A)$ maps to a loop in S^1 that is homotopic to the trivial loop, since it laps around and then backwards, so the homomorphism cannot be surjective, and hence no retract exists.

(d) If such a retract existed, we would have a surjective homomorphism from $\pi_1(D^2 \vee D^2) \cong 1 \to \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$.

(e) X deformation retracts onto S^1, so if such a retract existed to $S^1 \vee S^1$, there would be a surjective homomorphism from $\pi_1(S^1) \cong \mathbb{Z} \to \pi_1(S^1 \vee S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

(f) X deformation retracts to its central circle. Let γ be a loop around A. Composing this loop with a loop going around the central circle of X gives a loop that goes around the central circle twice. Hence the inclusion of A in X induces a homomorphism from $\mathbb{Z} \to 2\mathbb{Z}$ that restricts to the identity on $2\mathbb{Z}$, which is impossible, so no such retract exists. ■

Exercise 15. *If a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is null-homotopic.*

Proof. Let $f : X \to S^1$ be any map. Since $\pi_1(X)$ is finite, $f_*(\pi_1(X))$ is a finite subgroup of $\mathbb{Z}$ and hence trivial. By the Lifting Criterion, there exists a lift $\widetilde{f} : X \to \mathbb{R}$ of f. $\mathbb{R}$ is contractible, so $\widetilde{f}$ is null-homotopic by some homotopy h. But if $p : \mathbb{R} \to S^1$ is a covering map, then $p \circ h$ is a homotopy from f to a constant map. ■

Exercise 16. *For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\widetilde{X}$ abelian if it is normal and has an abelian deck transformation group. X has an abelian covering space that is a covering space of every other abelian covering space of X, and such a âĂŸuniversalâĂŹ abelian covering space is unique up to isomorphism. Below is a description of this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.*

Proof. The commutator subgroup $[\pi_1(X), \pi_1(X)]$ is a normal subgroup, and hence corresponds to a normal, path-connected covering space $\widetilde{X}$. The group of deck transformations of $\widetilde{X}$ is also abelian, since the quotient of a group by the commutator subgroup yields that groups abelianization.

Suppose there exists another covering map $q : \widehat{X} \to X$ with $q_*(\pi_1(\widehat{X}))$ normal and the quotient $\pi_1(X)/q_*(\pi_1(\widehat{X}))$ abelian. The commutator subgroup of $\pi_1(X)$ lies inside $q_*(\pi_1(\widehat{X}))$, so by the lifting criterion, the map $p : \widetilde{X} \to X$ lifts to a map $\widetilde{p} : \widetilde{X} \to \widehat{X}$. But the lift of a covering map is a covering map, so $\widetilde{X}$ covers $\widehat{X}$.

Now suppose actually that $\widehat{X}$ is also a universal abelian cover. Then both $\widetilde{p}$ and $\widehat{q}$ are lifts of the covering maps satisfying $p\widehat{q}\widetilde{p} = p$, so by the uniqueness of lifts, $\widehat{q}\widetilde{p}$ must be the identity on $\widetilde{X}$. A similar argument shows that $\widetilde{p}\widehat{q}$ is the identity on $\widehat{X}$, so we must have an isomorphism between $\widetilde{X}$ and $\widehat{X}$.

Concretely, if $X = S^1 \vee S^1$, then we are looking for a space whose group of deck transformations is the abelianization of $Z * Z$, which is $Z \times Z$. This would be some two dimensional lattice where horizontal movements correspond to one generator and vertical movements correspond to another generator. The case for $S^1 \times S^1 \times S^1$ should be a similar but three dimensional lattice. ■

> **Exercise 17.** *Given a covering space action of a group G on a path-connected, locally path-connected space X, then each subgroup $H < G$ determines a composition of covering spaces $X \to X/H \to X/G$. Then*
>
> *(a) Every path-connected covering space between X and X/G is isomorphic to X/H for some subgroup $H < G$.*
>
> *(b) Two such covering spaces X/H_1 and X/H_2 of X/G are isomorphic if and only if H_1 and H_2 are conjugate subgroups of G.*
>
> *(c) The covering space $X/H \to X/G$ is normal if and only if H is a normal subgroup of G, in which case the group of deck transformations of this cover is G/H.*

Proof. (a) Given a sequence $X \xrightarrow{p_1} Y \xrightarrow{p_2} X/G$, let $H = \{g \in G \mid p_1 \circ g = p_1\}$. p_1 descends to a map $\widetilde{p}_1 : X/H \to Y$ since p_1 is constant on equivalence classes. That is, if $x_1 \sim_H x_2$, then $h(x_1) = x_2$ for some $h \in H$. But then $p_1(x_1) = p_1 h(x_1) = p_1(x_2)$. Define a map $q : Y \to X/H$ by $q(y) = [x]_H$, where x is an element in the fiber of y under p_1. This map is well-defined since if $p_1(x) = p_1(x') = y$, then $p_2p_1(x) = p_2p_1(x')$, so $x \sim_G x'$. But then $p_1 \circ g(x) = p_1(x)$, so $p_1 \circ g = p_1$, and hence $g \in H$. So Y and X/H are isomorphic since $\widetilde{p}_1$ is a continuous bijection preserving the covering.

(b) $(\implies)$ Given an isomorphism $f : X/H_1 \to X/H_2$. Since f is an isomorphism, we have $qf = p$ where p and q are the covering maps from $X/H_1 \to X/G$ and $X/H_2 \to X/G$. If $[x]_{H_1} \mapsto [x']_{H_2}$ under f, then there exists g such that $gx = x'$ since both $qf = p$.

$(\impliedby)$ Let g be such that $gH_1g^{-1} = H_2$. Define the map $f : [x]_{H_1} \mapsto [gx]_{H_2}$. This map is well defined since if $h(x) = x'$, then there exists h' such that $h'gx = gx'$, namely $h' = ghg^{-1}$. The map is a bijection since its inverse is given by the map generated by g^{-1}. Given an open set U in X/H_2, can take the inverse image under the natural projection to get an open set in X. But the taking the inverse isomorphism g^{-1} and projecting back onto X/H_1 gives an open set since the projection map is open, equivalent to $f^{-1}(U)$. ■

Exercise 18. *The complement of a finite set of points in $\mathbb{R}^n$ is simply connected if $n \geqslant 3$.*

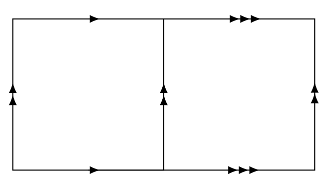
Proof. Let $S = \{x_1, x_2, \dots, x_m\}$ be a finite set of points in $\mathbb{R}^n$. Let $2\epsilon = \min_{i \neq j} |x_i - x_j|$ for $i \in \mathbb{N}$ such that $0 < i < m + 1$. Connect $B_\epsilon(x_i)$ to $B_\epsilon(x_{i+1})$ by a path for every $i \in \mathbb{N}$ such that $0 < i < m$. Then the complement of S deformation retracts to the boundaries of each ball together with the paths between them (first deformation retract onto a ball containing the entire system, then deflate the ball). But this is homotopy equivalent to $\bigvee_{i=1}^m S^{n-1}$. Each S^{n-1} is a CW complex, so $\pi_1(\bigvee_{i=1}^m S^{n-1}) = \ast_{i=1}^m \pi_1(S^{n-1}) = \ast_{i=1}^m 0 = 0$ by Van-Kampen. ■

Exercise 19. *Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. $\pi_1(\mathbb{R}^3 - X) = \mathbb{Z}^{*(2n-1)}$.*

Proof. $\mathbb{R}^3 - X$ deformation retracts onto S^2 minus $2n$ points, where the $2n$ points are the intersections of S^2 with the lines and the deformation retract is the usual one onto the unit sphere along rays from the origin. But by the stereographic projection, $S^2 - (S^2 \cap X)$ is homeomorphic to $\mathbb{R}^2$ minus $2n - 1$ points. $\mathbb{R}^2$ minus $2n - 1$ points has a bouquet of $2n - 1$ circles as a deformation retract, so its fundamental group is $Z^{*(2n-1)}$. ■

Exercise 20. *The fundamental group obtained from two tori by identifying $S^1 \times \{x_0\}$ in one torus to $S^1 \times \{x_0\}$ in the other torus is $(\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$.*

Proof. The identification space is



This gives a presentation for the group as $G = \langle a, b, c \mid abb^{-1}a^{-1} = e,\ bcc^{-1}b^{-1} = e \rangle$. We have the free group on three generators, where one of the generators commutes with both of the others. This gives $(\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$.

The space is just $(S^1 \vee S^1) \times S^1$. To see this, place one torus inside the hole of the other, and identify the inner circle of the outer torus with the outer circle of the inner torus. This gives $\pi_1((S^1 \vee S^1) \times S^1) = \pi_1(S^1 \vee S^1) \times \pi_1(S^1) = (\pi_1(S^1) * \pi_1(S^1)) \times \pi_1(S^1) = (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$. ■

Exercise 21. *$\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.*

Proof. Consider $(x_0, x_0) \in \mathbb{R}^2 - \mathbb{Q}^2$. For any $(x, x) \in \mathbb{R}^2 - \mathbb{Q}^2$ with $x_0 < x$, let γ_x be the box path from $(x_0, x_0) \to (x, x_0) \to (x, x) \to (x_0, x) \to (x_0, x_0)$. For rational r such that $x_0 < r < x$, (r, r) is enclosed by γ_x, so γ_x is not trivial. It remains to show γ_x is not homotopic to $\gamma_{x'}$ for $x \neq x'$.

Without loss of generality, assume $x < x'$. Again there exists rational r with $x < r < x'$. Consider the inclusion map $i : (\mathbb{R}^2 - \mathbb{Q}^2) \hookrightarrow (\mathbb{R}^2 - \{(r, r)\})$. $[\gamma_x]$ is in the kernel of i_*, but $\gamma_{x'}$ isn't, so γ_x and $\gamma_{x'}$ are not homotopic. ■

Exercise 22. *Of the following, only (d) is not a category.*

(a) Objects are finite sets, morphisms are injective maps of sets.
(b) Objects are sets, morphisms are surjective maps of sets.
(c) Objects are abelian groups, morphisms are isomorphisms of abelian groups.
(d) Objects are sets, morphisms are maps of sets which are not surjective.
(e) Objects are topological spaces, morphisms are homeomorphisms.

Proof. (a) Composition of injective maps is injective. Composition is associative. The identity is an injection.

(b) Composition of surjective maps is surjective. Composition is associative. The identity is a surjection.

(c) Composition of isomorphisms is an isomorphism and composition is associative. The identity is an isomorphism.

(d) The identity map is surjective, so the identity is not a morphism, so this can't be a category.

(e) Composition of homeomorphisms is a homeomorphism and composition is associative. The identity is a homeomorphism. ∎

Exercise 23. *Below are examples of categories with*
(a) One object and four morphisms.
(b) Two objects and five morphisms.

Proof. (a) We can regard $\mathbb{Z}_4$ as a category with one object and four morphisms. The object is the underlying set, and the morphisms are the maps given by adding by an element.

(b)
$$\begin{tikzcd} A \arrow[loop above] \arrow[loop left] \arrow[loop below] \arrow[r] & B \arrow[loop above] \end{tikzcd}$$

∎

Exercise 24. *The simplest possible category is the empty category* $\mathbf{0}$*, consisting of no objects and no morphisms. Given another category* $\mathbf{C}$*, there is a unique functor* $F : \mathbf{0} \to \mathbf{C}$*, taking nothing nowhere. By definition,* $\operatorname{colim} F$ *is called an initial object of* $\mathbf{C}$*, if it exists.*

(a) Any two initial objects in a category are uniquely isomorphic.

(b) Below is a description of which of the following categories have initial objects, and what they are: Set, Gp, Top, Top, the category of fields with field homomorphisms, the category of infinite-dimensional vector spaces over a given field with linear maps, the category of small categories with functors Cat.*

(c) Below is a definition of the notion of a terminal object in a category, and a description of the terminal objects (if they exist) in the previous categories.

Proof. (a) Suppose 0 and $0'$ are initial objects in C. Then there exist unique morphisms $f : 0 \to 0'$ and $g : 0' \to 0$. Composing, we have fg and gf as unique morphisms $0' \to 0'$ and $0 \to 0$. But the respective identity morphisms are such morphisms, so they must be the identity morphisms by uniqueness. Hence 0 and $0'$ are isomorphic, and the isomorphism is unique.

(b) An initial object is an object such that for every object in the category, there exists precisely one morphism from the initial object to that object.

Set has the empty set as its initial object, since theres only one map from the empty set to any other set.

Gp has the trivial group as its initial object since there's only one homomorphism from the trivial group to any other group, sending the identity to the identity.

Top has the empty set as its initial object since theres only one function from the empty set to any other topological space.

$\mathbf{Top}_*$ has the one point space $*$ as its initial object, since the point must go to the base point of any other space under any continuous map.

Field has no initial object.

$\mathbf{Vec}_\infty$ has no initial object.

Cat has the empty category as its initial object since theres only one map from the empty category to any other category taking nothing nowhere.

(c) A terminal object is an object such that for every object in the category, there is a unique morphism from that object to the terminal object.

Set has any one element set as a terminal object, where the only maps to the one element set are the maps sending everything to that set.

Gp has the trivial group as its terminal object as well, since the only map to the trivial group sends everything to the identity.

Top has $*$ as a terminal object since any map to $*$ sends everything to $*$.

$\mathbf{Top}_*$ has the one point space $*$ as its terminal object since any map sends everything to $*$.

Field has no terminal object.

$\mathbf{Vec}_\infty$ has no terminal object.

Cat has the category with one object and one morphism as its terminal object, since any functor must send everything to that one object.

■

Exercise 25. *A natural isomorphism is a natural transformation α, say between two functors $F, G : \mathbf{C} \to \mathbf{D}$, such that α_X is an isomorphism for each $X \in Ob(\mathbf{C})$. Two categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that there are natural isomorphisms $\epsilon : FG \to id_{\mathbf{D}}$ and $\eta : id_{\mathbf{C}} \to GF$. Let X be a topological space and $x \in X$. Regard $\pi_1(X, x)$, a group, as a category with one element x. Let $\Pi_1(X)$ be the fundamental groupoid of X: its objects are points of X and its morphisms from x to y are homotopy classes of paths (with fixed endpoint) from x to y. There is an evident âĂIJinclusion functorâĂİ $J : \pi_1(X, x) \to \Pi_1(X)$.*

If X is path-connected, J is an equivalence of categories.

Proof. Following the proof by Peter May, define the inverse functor $F : \Pi_1(X) \to \pi_1(X, x)$ where $F(A)$ is an object isomorphic to A in $\pi_1(X, x)$. choosing an isomorphism $\alpha_A : A \to F(A)$ and mapping morphisms $f : A \to B$ as $F(f) = \alpha_B \circ f \circ \alpha_A^{-1} : F(A) \to F(B)$. Let α_A be the identity if $A \in \pi_1(X, x)$. Then $FJ = \text{id}$ and $\alpha_A : \text{id} \to JF$ is a natural isomorphism. ■

Exercise 26. *Given two maps of topological spaces $f : X \to Z$ and $g : Y \to Z$, let $X \times_Z Y$, the pullback or fiber product of f and g, be defined as the set*

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

equipped with the subspace topology inherited from the product, together with the âĂIJobviousâĂİ maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ formed by composing inclusion and projection onto one of the factors. The pullback is a limit in **Top** *over the diagram $X \to Z \leftarrow Y$.*

Proof. Let T, x, y be a cone over the same diagram. It suffices to show there exists a morphism u from T to the pullback such that the following diagram commutes.

$$\begin{tikzcd}
T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Let $u(t) = (x(t), y(t))$. This is well-defined since $g(y(t)) = f(x(t))$, so $u(t) \in X \times_Z Y$ for all $t \in T$. u is continuous since the coordinate functions are continuous and the pullback has the subspace topology inherited from the product topology. More explicitly,the product topology has rectangular open sets as a basis. The preimage of any basis element under u is the intersection of the preimages under x and y, which is open as it is the intersection of two open sets. ■

Exercise 27. *Given two maps of topological spaces $f : Z \to X$ and $g : Z \to Y$, the pushout of f and g is defined as the quotient space*

$$X \coprod_Z Y = X \coprod Y / \sim$$

where $\sim$ is the equivalence relation generated by $f(z) \sim g(z)$ for all $z \in Z$, together with the "obvious" maps $X \to X \coprod_Z Y$ and $Y \to X \coprod_Z Y$ formed by composing inclusion into $X \coprod Y$ and the quotient map. The pushout is a colimit in **Top** *over the diagram $X \leftarrow Z \to Y$.*

Proof. Let T, j_1, j_2 be a cocone over the same diagram. It suffices to show there exists a morphism u from the pushout to T such that the following diagram commutes.

$$\begin{tikzcd}
T & & \\
& X \coprod_Z Y \arrow[ul, dotted, "u"'] & Y \arrow[l, "i_2"] \arrow[ull, bend right, "j_2"'] \\
& X \arrow[u, "i_1"] \arrow[uul, bend left, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

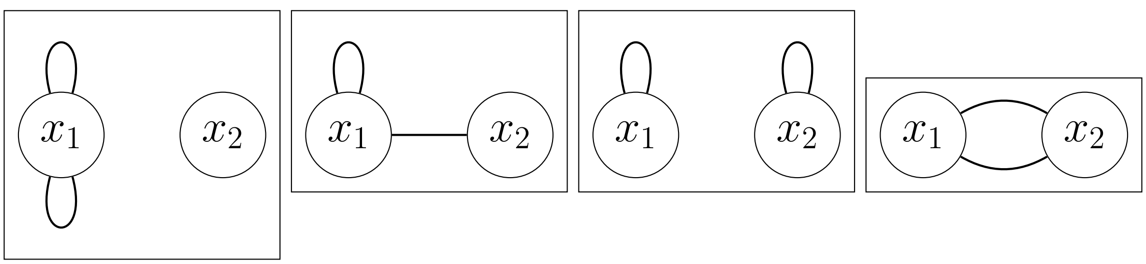
Let $u(w) = \begin{cases} j_1(w) & w \in i_1(X) \\ j_2(w) & w \in i_2(Y) \end{cases}$. Both j_1 and j_2 are continuous and agree on $i_1(X) \cap i_2(Y)$ as $j_1 \circ f = j_2 \circ g$. Both $i_1(X)$ and $i_2(Y)$ are closed, so by the gluing lemma, u is continuous. ■

Exercise 28. *The mapping cylinder M_f of a map $f : X \to Y$ is a colimit over the diagram $X \times I \leftarrow X \to Y$, where $X \to X \times I$ is the natural map identifying X with $X \times \{0\}$.*

Proof. M_f is the pushout $X \times I \coprod_X Y$ with the maps $f : X \to Y$ and $i : X \to X \times I$ identifying X to $X \times \{0\}$, and hence by the previous proposition, it is a colimit over the diagram $X \times I \leftarrow X \to Y$. ■

Exercise 29. *All CW complexes with two 0-cells and two 1-cells up to*
(a) homeomorphism are in one of the four classes described below.
(b) homotopy equivalence are in one of the three classes described below.

Proof. (a) We have two maps $f : \{a, b\} \to \{x_1, x_2\}$ and $g : \{a', b'\} \to \{x_1, x_2\}$, giving 16 possible constructions. But up to relabeling of nodes and changing directions of edges (homeomorphism), we only have



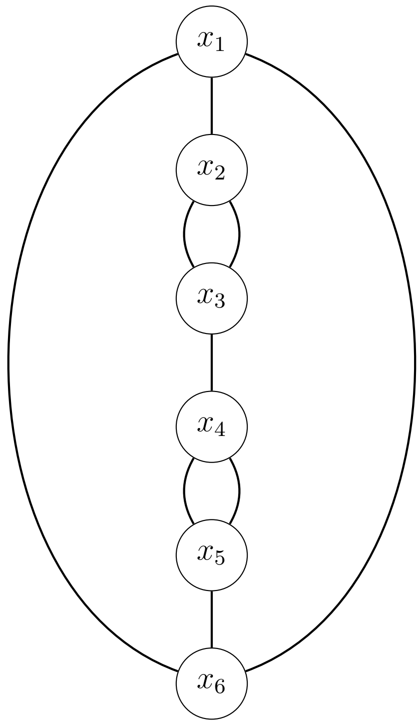
(b) The second box from the left is homotopy equivalent to the last box from the left by a contraction of the x_2 node to the x_1 node.

■

Exercise 30. *(a) The 'square lattice' is a CW complex homeomorphic to $\mathbb{R}^2$.*
(b) The following diagram is a CW complex homeomorphic to a 2-disk with two smaller open 2-disks removed.

Proof. (a) Let the 0-skeleton be $\mathbb{Z} \times \mathbb{Z}$. To construct the 1-skeleton, connect nodes a distance one apart to one-another by the path of distance one. To construct the 2-skeleton, fill in each square $(i, j) \to (i + 1, j) \to (i + 1, j + 1) \to (i, j + 1) \to (i, j)$.

(b) The one skeleton is given by



Gluing a disk to the large left region and a disk to the large right region gives the desired disk with two smaller open disks removed. ■

Exercise 31. *$\mathbb{R}P^n - \{x\}$, where $x \in \mathbb{R}P^n$ is any point, is homotopy equivalent to $\mathbb{R}P^{n-1}$.*

Proof. $\mathbb{R}P^n$ is $S^n/(v \sim -v)$, but this is equivalent to D^n with antipodes on ∂D^n identified. ∂D^n with antipodes identified is just $\mathbb{R}P^{n-1}$, so the real projective space of dimension n can be constructed by $\mathbb{R}^{n-1} \bigcup_f D^n$ where the attaching map $f : S^{n-1} \to \mathbb{R}P^{n-1}$ is the quotient projection. If the point to be removed lies on the boundary of D^n, use a homeomorphism to move it to the interior. Then D^n minus an interior point deformation retracts onto its boundary, so $\mathbb{R}P^n$ minus a point deformation retracts to $\mathbb{R}P^{n-1}$, and hence they are homotopy equivalent. ■

Exercise 32. *(a) The mapping cylinder of every map $f : S^1 \to S^1$ is a CW complex.*
(b) The follwoing is a 2 dimensional CW complex that contains both an annulus $S^1 \times I$ and a MÃűbius band as deformation retracts.

Proof. (a) M_f in this instance can be constructed explicitly as a CW complex. Let the 0-skeleton be two points, θ_0 and $f(\theta_0)$. Attach a 1-cell to θ_0 and a 1-cell to $f(\theta_0)$, forming a loop at each point. Attach a 1-cell connecting θ_0 to $f(\theta_0)$. Attach a 2-cell along the path going along the loop at θ_0, along the 1-cell joining θ_0 to $f(\theta_0)$, along the image of the loop at s_0 under f, then back along the path joining the 0-cells.

(b) Place CW structures on the annulus and the MÃűbius band, and identify the central circle of the MÃűbius band with $S^1 \times \{0\}$. But then this CW complex retracts onto both the annulus and the MÃűbius band, by using the retraction of the MÃűbius band to its central circle and the retract of the annulus to $S^1 \times \{0\}$, respectively. ■

Exercise 33. *Below is a description of which of the following inclusions are cofibrations.*

(a) $\{x\} \hookrightarrow S^n$*, where* $x \in S^n$ *is any point.*

(b) $(0,1] \hookrightarrow [0,1]$.

(c) $\mathbb{Z} \hookrightarrow \mathbb{R}$.

(d) $\mathbb{Q} \hookrightarrow \mathbb{R}$.

(e) $X \hookrightarrow CX$*, where* $CX = (X \times I)/(X \times \{0\})$ *is the cone of* X *and the inclusion sends* $X \to X \times \{1\}$.

Proof. For the following, consider the diagram

$$\begin{tikzcd}
Y & X \arrow[l, "\widetilde{f}_0"'] \arrow[dl, dotted, "\widetilde{f}"] \\
Y^I \arrow[u, two heads, "p_0"] & A \arrow[l, "f"] \arrow[u, "i"']
\end{tikzcd}$$

A cofibration exists if and only if there is a retraction $X \times I$ to $A \times I \cup X \times \{0\}$. But if $A \subset X$, then the cofibration, if it exists, must be the inclusion, since a cofibration is injective and is a homeomorphism onto its image.

(a) This is a cofibration since there exists a retraction from $S^n \times I$ to $(I \times \{x\} \cup (X \times \{0\})$.

(b) A cofibration cannot exist since there is no retraction from I^2 to $((0,1] \times I) \cup (I \times \{0\})$.

(c) This is a cofibration since there is a retraction from $\mathbb{R} \times I$ to $(\mathbb{Z} \times I) \cup (\mathbb{R} \times \{0\})$.

(d) A cofibration cannot exist since there is nor etraction from $\mathbb{R} \times I$ to $(\mathbb{Q} \times I) \cup (\mathbb{R} \times \{0\})$.

(e) This is a cofibration since there exists a retract from $CX \times I$ to $(X \times I) \cup (CX \times \{0\})$. ■

Exercise 34. *If* $f : X \to Y$ *is a (co)fibration, and* $g : Y \to Z$ *is a (co)fibration, then* $g \circ f : X \to Z$ *is a (co)fibration.*

Proof. Let W be any topological space, $h : W \times I \to Z$ be any continuous map, i be the map sending W to $W \times \{0\}$, and $\widetilde{h}_0$ be such that $h \circ i = g \circ f \circ \widetilde{h}_0$. $g \circ f$ is a fibration iff there exists a function u such that Diagram A commutes. Well, g is a fibration, so using $f \circ \widetilde{h}_0$, there exists a lift $\widehat{h}$ such that Diagram B commutes. f is another fibration, so there must exist a lift $\widetilde{h}$ such that $\widetilde{h} \circ i = \widetilde{h}_0$. Then letting $u = \widetilde{h}$ gives commutative Diagram C, so $g \circ f$ is a fibration.

Diagram A

$$\begin{tikzcd}
W \arrow[r, "\widetilde{h}_0"] \arrow[dd, hook, "i"'] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
W \times I \arrow[r, "h"] \arrow[uur, dashed, "u"] & Z
\end{tikzcd}$$

Diagram B

$$\begin{tikzcd}
W \arrow[r, "\widetilde{h}_0"] \arrow[dd, hook, "i"'] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
W \times I \arrow[r, "h"] \arrow[uur, dashed, "u"] \arrow[ur, bend left, "\widehat{h}"] & Z
\end{tikzcd}$$

Diagram C

$$\begin{tikzcd}
W \arrow[r, "\widetilde{h}_0"] \arrow[dd, hook, "i"'] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
W \times I \arrow[r, "h"] \arrow[uur, bend left, "\widetilde{h}"] \arrow[ur, bend left, "\widehat{h}"] & Z
\end{tikzcd}$$

Reversing arrows in the diagrams gives a proof that cofibrations are closed under composition. ■

Exercise 35. *If $*$ is the one point space, then any map $f : X \to *$ is a fibration.*

Proof. Let Z be any topological space, i be the map sending Z to $Z \times \{0\}$, and let h and $\widetilde{h}_0$ be continuous maps such that $h \circ i = f \circ \widetilde{h}_0$. Then f is a fibration iff there exists a map u such that the following diagram commutes

$$\begin{tikzcd} Z \arrow[r, "\widetilde{h}_0"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "h"] \arrow[ur, dashed, "u"] & * \end{tikzcd}$$

Let $u(z,t) = \widetilde{h}_0(z)$. Then $u \circ i = \widetilde{h}_0$, and $f \circ u = h$ since any map $X \to *$ is constant, so the diagram commutes. ∎

Exercise 36. *A map $p : E \to B$ is a fibration iff the map $\pi : E^I \to E_p, \pi(\gamma) = (\gamma(0), p\gamma)$, has a section, that is, a map $s : E_p \to E^I$ such that $\pi s = 1$.*

Proof. ($\implies$) Consider the following diagram

$$\begin{tikzcd} E_p \arrow[r, "(e,\gamma)\mapsto e"] \arrow[d, hook, "E_p\times\{0\}"'] & E \arrow[d, "p"] \\ E_p \times I \arrow[r, "(e,\gamma,t)\mapsto\gamma(t)"'] \arrow[ur, dashed, "s"] & B \end{tikzcd}$$

The diagram commutes since $\gamma(0) = p(e)$, so the lift s exists since p is a fibration. We have $\pi(s(e,\gamma)) = (s(e,\gamma)(0), ps(e,\gamma))$. But by the diagram, $s(e,\gamma)(0) = e$ and $ps(e,\gamma) = \gamma$, so $\pi s = 1$.

($\impliedby$) It suffices to show there exists $\widetilde{f}$ making

$$\begin{tikzcd} X \arrow[r, "\widetilde{f}_0"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & B \end{tikzcd}$$

commute. Since $\pi s = 1$, $\pi(s(e,\gamma)) = (s(e,\gamma)(0), ps(e,\gamma)) = (e,\gamma)$ so $s(e,\gamma)(0) = e$ and $(ps)(e,\gamma) = \gamma$. Let $\widetilde{f}(x,t) = (s(\widetilde{f}_0(x), f_t(x)))(t)$. This is a valid lift since $\widetilde{f}(x,t) = (s(\widetilde{f}_0(x), f_0(x)))(0) = \widetilde{f}_0(x)$ as $s(e,\gamma)(0) = e$. It is continuous since $s, \widetilde{f}_0$ and f_t are continuous. The diagram commutes since $p\widetilde{f} = p(s(\widetilde{f}_0(x), f_t(x))(t)) = f(x,t)$ as $(ps)(e,\gamma) = \gamma$. ∎

Exercise 37. *A linear projection of a 2-simplex onto one of its edges is a fibration but not a fiber bundle.*

Proof. Let T be any 2 simplex in $\mathbb{R}^2$. Project onto a side where both incident angles are less than $\pi/2$, which must exist since otherwise the angles would sum to more than π. Then the fiber of an endpoint belonging to that edge is a single point, while the fiber of any point that's not an endpoint is a line, so the fibers are not homeomorphic, and thus the projection is not a fiber bundle.

It remains to show the projection is a fibration. We have $\pi : T^I \to T_p$ as in the previous proposition given by $\pi(\gamma) = (\gamma(0), p\gamma)$ where $p : T \to T_e$ is the projection onto the edge. It suffices to construct a section such that $\pi s = 1$. The section is given explicitly by using a path that is mimics the path on the edge, except it slides along edges if the path it tries to mimic goes outside the triangle. ∎

Exercise 38. *If X and Y are pointed topological spaces, then $\langle \Sigma X, Y\rangle = \langle X, \Omega Y\rangle$.*

Proof. Let $\phi : F(X, \Omega Y) \to F(\Sigma X, Y)$ be given by $\phi(h)(x, s) = h(x)(s)$. ϕ is well-defined since $\phi(h)(x, s_0) = h(x)(s_0) = y_0$ and $\phi(h)(x_0, s) = h(x_0)(s) = y_0$, where y_0 denotes the loop staying put at y_0 for all $s \in S^1$. Let $\phi^{-1}(g)(x)(s) = g(x, s)$. Then we have $\phi(\phi^{-1}(g))(x, s) = \phi^{-1}(g)(x)(s) = g(x, s)$ and $\phi^{-1}(\phi(h))(x)(s) = \phi(h)(x, s) = h(x, s)$. Hence ϕ is a bijection. But ϕ is a restriction of the usual currying function, and since S^1 is locally compact Hausdorff, so the currying function is continuous, and in particular ϕ. The inverse is continuous since it is the un-currying function.

It remains to show that ϕ descends on based homotopy equivalence classes. But if H is a homotopy between $h, h' \in F(X, \Omega Y)$, then $\phi \circ H$ is a homotopy between ϕh and $\phi h'$. The composition is a valid homotopy since ϕ and H are continuous. ∎

Exercise 39. *Suppose that X is a CW complex that is an increasing union of subcomplexes $X_1 \subset X_2 \subset \dots$ such that each inclusion $X_j \hookrightarrow X_{j+1}$ is nullhomotopic. Then X is contractible.*

Proof. Each pair (X, X_j) satisfies the homotopy extension property since the X_j are subcomplexes, and thus each null-homotopy $h_j : X_j \times I \to X_{j+1}$ extends to a homotopy $\widetilde{h}_j : X \times I \to X$. As in the proof of inclusion of subcomplexes being cofibrations, define $h : X \times I \to X$ to be the composition of all of the homotopy extensions. This gives a contraction of X. ∎

Exercise 40. *Given a pointed space (X, x_0), view $\sum X$ as $X \times I/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)$. Then the inclusion $i : X \hookrightarrow \sum X$ given by $x \mapsto (x, 1/2)$ is nullhomotopic.*

Proof. Let $h(x, t) = (x, \frac{1}{2}(1 + t))$. Then $h(x, 0) = (x, \frac{1}{2})$ and $h(x, 1) = (x, 1)$. But under the equivalence, all points $(x, 1)$ are identified to a single point. ∎

Exercise 41. *If X is a pointed CW complex, then the infinite suspension*

$$\Sigma^\infty(X) = \bigcup_{j \geqslant 1} \Sigma^j(X)$$

with inclusions as in the previous proposition is contractible. Since Σ^∞ is a functor, this gives a way of making arbitrary CW complexes contractible. In particular, S^∞ is contractible.

Proof. $\Sigma^\infty(X)$ is an increasing union of subcomplexes $X \subset \Sigma X \subset \Sigma^2 X \subset \dots$. Each inclusion $i : \Sigma^j X \hookrightarrow \Sigma^{j+1}$ is null-homotopic by **Proposition 7**, so by **Proposition 6**, the infinite suspension is contractible. ∎

Exercise 42. *If a diagram*

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\ \downarrow{\scriptstyle \widetilde{j}} & & \downarrow{\scriptstyle \widetilde{l}} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \widetilde{m}} & & \downarrow{\scriptstyle \widetilde{n}} \\ A' & \xrightarrow{o} & B' & \xrightarrow{p} & C' & \xrightarrow{q} & D' & \xrightarrow{r} & E' \end{array}$$

of homomorphisms of abelian groups with exact rows commutes, and all vertical maps except the middle are isomorphisms, then the middle map is injective.

Proof. Let 0 denote the identity element in a group. Let c be an element in the kernel of k. Any homomorphism maps $0 \mapsto 0$, so $q(k(c)) = 0$. Since the diagram commutes, $q(k(c)) = \widetilde{m}(h(c)) = 0$. But $\widetilde{m}$ is an isomorphism, so $h(c) = 0$. By exactness of the first row, there exists an element $b \in B$ such that $g(b) = c$. Since the diagram commutes, $p(\widetilde{l}(b)) = k(g(b)) = k(c) = 0$. By exactness of the second row, there exists a' such that $o(a') = \widetilde{l}(b)$. $\widetilde{j}$ is an isomorphism, so there exists $a \in A$ such that $\widetilde{j}(a) = a'$. The diagram commutes, so $\widetilde{l}f(a)) = o(\widetilde{j}(a)) = o(a') = \widetilde{l}(b)$. Since $\widetilde{j}$ is injective, $f(a) = b$. But $g(f(a)) = g(b) = c$ and $g(f(a)) = 0$ by exactness of the diagram, so $\ker k = 0$ and hence k is injective.

Assumptions: maps are homomorphisms, rows are exact, commutativity of the diagram, and $\widetilde{j}$ and $\widetilde{m}$ are isomorphisms. ∎

Exercise 43. *If $p : (E, e_0) \to (B, b_0)$ is a based fibration of based spaces, then the inclusion $\phi : F = p^{-1}(b) \to N_p$ is a based homotopy equivalence.*

Proof. Explicitly, $N_p = \{(e, \gamma) \in E \times B^I : \gamma(0) = p(e), \gamma(1) = b_0\}$. Let $h : N_p \times I \to B$ be given by $h_t(e, \gamma) = \gamma(t)$. Let $\widetilde{h}_0 : N_p \to E$ be given by $\widetilde{h}_0(e, \gamma) = e$. By the homotopy lifting property, there exists a lift $\widetilde{h} : N_p \times I \to E$ such that

$$\begin{tikzcd} N_p \arrow[r, "\widetilde{h}_0"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ N_p \times I \arrow[r, "h"] \arrow[ur, "\widetilde{h}"] & B \end{tikzcd}$$

commutes. Let $H : N_p \times I \to N_p$ be given by $H_t(e, \gamma) = (\widetilde{h}_t(e, \gamma), \gamma|_{[t,1]})$. Note that H_t is fiber preserving since the endpoints of the paths are unchanged.

The map $e \mapsto (e, c_{p(e)})$ is a homeomorphism between E and its image in N_p, call it E'. Note that $H_1(e, \gamma) = (\widetilde{h}_1(e, \gamma), c_{\gamma(1)})$, and since by commutativity of the diagram, $p(\widetilde{h}_1(e, \gamma)) = h_1(\gamma) = \gamma(1)$, $H_1(N_p) \subset E' \cong E$. We can thus regard H_1 as a map $N_p \to E$. Then $\phi H_1 = H_1$. Additionally, we have that $H_t(E) \subset E' \cong E$ for all t.

Note that $H_0 = \mathrm{id}$, so that by H_t, we have that ϕH_1 is based homotopy equivalent to the identity. Additionally, $H_1 \phi$ is based homotopy equivalent to the identity by $H_t|_E$, so ϕ is a based homotopy equivalence of fibers. ∎

Exercise 44. *Let $f : X \to Y$ be a map of pointed topological spaces and let $\pi : N_f \to X$ be the projection. Then π is always a fibration, and the natural inclusion $i : \Omega Y \to N_\pi$ is a homotopy equivalence.*

Proof. Let

$$\begin{tikzcd} Z \arrow[r, "\widetilde{h}_0"] \arrow[d, hook, "i"'] & N_f \arrow[d, "\pi"] \\ Z \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & X \end{tikzcd}$$

be a commutative diagram without $\widetilde{h}$. π is a fibration if and only if there exists $\widetilde{h}$ such that the diagram still commutes.

If such a lift exists, since $\pi \circ \widetilde{h} = h$, the first coordinate of the lift must be $\widetilde{h}_t(z)_1 = h_t(z)$. To be well defined, the second coordinate of the lift must be a path from $f(h_t(z))$ to y_0, where y_0 is the basepoint of Y. Interpret and reparametrize $h_t(z)$ as a path from $h_0(z)$ to $h_t(z)$. Then, a path from $f(h_t(z))$ to $f(h_0(z))$ is given by $\gamma_z(t) = f(\widehat{h}_t(z))$, where $\widehat{h}_t(z)$ denotes the inverse path to $h_t(z)$. Concatenate this path with the second coordinate of the initial lift $\widetilde{h}_0(z)_2$. This is well-defined since $\widetilde{h}_0(z)_2$ must be a path from $f(\widetilde{h}_0(z)_1)$ to y_0 be definition of N_f, and by commutativity of the diagram, $h_0(z) = \pi(\widetilde{h}_0(z))$, so $h_0(z) = \widetilde{h}_0(z)_1$. Hence the lift $\widetilde{h}$ exists as is given explicitly by $\widetilde{h}_t(z) = (h_t(z), \widetilde{h}_0(z)_2 * \gamma_z(t))$. ∎

Exercise 45. *For $n \in \mathbb{N} \cup \{\infty\}$, $\pi_i(S^n) \cong \pi_i(\mathbb{R}P^n)$ for all $i > 1$.*

Proof. Pick arbitrary basepoints $s_0 \in S^n$ and $p_0 \in \mathbb{R}P^n$. Let $\pi : (S^n, s_0) \to (\mathbb{R}P^n, p_0)$ be the basepoint preserving natural projection identifying antipodal points. π is a covering map by **Proposition 1.40** in Hatcher, so therefore it is also a fibration. Identify S^0 as $\pi^{-1}(\{p_0\})$ with an arbitrary basepoint s_0'. Let $i : S_0 \to S^n$ be the basepoint preserving inclusion. Then we have the long exact sequence of homotopy groups

$$\ldots \to \pi_i(S^0, s_0') \to \pi_i(S^n, s_0) \to \pi_i(\mathbb{R}P^n, p_0) \to \pi_{i-1}(S^0, s_0') \to \ldots$$

which follows from the long exact sequence of a fibration. For $i > 1$, $\pi_i(S^0, s_0')$ is trivial, so the short exact sequence

$$0 \xrightarrow{f} \pi_i(S^n, s_0) \xrightarrow{g} \pi_i(\mathbb{R}P^n, p_0) \xrightarrow{h} 0$$

is valid for all $i > 1$. g is injective since $\ker g = f(\{0\}) = 0$, and it is surjective since its image is $\ker h$. Therefore, $\pi_i(S^n, s_0) \cong \pi_i(\mathbb{R}P^n, p_0)$. The proposition follows since both spaces are path-connected.

This proposition holds also for S^∞ and $\mathbb{R}P^\infty$ since S^∞ is a double cover of $\mathbb{R}P^\infty$. ∎

Exercise 46. *Let $m, n \in \mathbb{Z}_{>1} \cup \{\infty\}$ and let $X = \mathbb{R}P^m \times S^n$ and $Y = \mathbb{R}P^n \times S^m$ (pick basepoints arbitrarily). Then $\pi_i(X) \cong \pi_i(Y)$ for all $i \geqslant 0$.*

Proof. Real projective spaces and spheres of dimension greater than one are path-connected, so by **Proposition 4.2** in Hatcher, $\pi_i(\mathbb{R}P^m \times S^n) \cong \pi_i(\mathbb{R}P^m) \times \pi_i(S^n)$ and $\pi_i(\mathbb{R}P^n \times S^m) = \pi_i(\mathbb{R}P^n) \times \pi_i(S^m)$.

The case $i = 0$ follows since both spaces are path-connected.

By **1B.3** in Hatcher, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \in \mathbb{Z}_{>1} \cup \{\infty\}$. Hence, for $i = 1$, $\pi_1(\mathbb{R}P^m \times S^n) \cong \pi_1(\mathbb{R}P^m) \times \pi_1(S^n) \cong \mathbb{Z}/2\mathbb{Z} \times 0 \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(\mathbb{R}P^n \times S^m) = \pi_1(\mathbb{R}P^n) \times \pi_i(S^m) \cong \mathbb{Z}/2\mathbb{Z} \times 0 \cong \mathbb{Z}/2\mathbb{Z}$.

The case $i > 1$ follows by the previous proposition. ∎

Exercise 47. *Let*

$$\dots \to C_{n+1} \xrightarrow{g_{n+1}} A_n \xrightarrow{i_n} B_n \xrightarrow{f_n} C_n \xrightarrow{g_n} A_{n-1} \xrightarrow{i_{n-1}} B_{n-1} \to \dots$$

be a long exact sequence of abelian groups, with every third arrow i_n injective. Then the short sequence

$$0 \xrightarrow{g} A_n \xrightarrow{i_n} B_n \xrightarrow{f_n} C_n \xrightarrow{h} 0$$

is exact.

Proof. i_n is an injective homomorphism, so $\ker i_n = 0 = g(\{0\})$. $\operatorname{im} i_n = \ker f_n$ since the long exact sequence is exact. Additionally, the exactness of the long sequence gives $\operatorname{im} g_n = \ker i_{n-1}$, which is trivial since i_{n-1} is injective. Then $\operatorname{im} g_n = 0$ gives $\ker g_n = C_n$. By exactness of the long sequence, $\operatorname{im} f_n = \ker g_n = C_n$. Hence $\operatorname{im} f = C_n = \ker h$, so the short sequence is exact. ∎

Exercise 48. *If*

$$0 \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{h} C \xrightarrow{j} 0$$

is a short exact sequence of abelian groups, and there exists a map $g : B \to A$ such that $gf = id_A$, then $B \cong A \oplus C$.

Proof. Let $\phi : B \to A \oplus C$ be given by $\phi(b) = (g(b), h(b))$. ϕ is a homomorphism since its coordinate functions are homomorphisms.

Suppose $b \in \ker \phi$. Then $g(b) = 0$ and $h(b) = 0$. $b \in \ker h$, hence there exists $a \in A$ such that $f(a) = b$ by exactness of the sequence. This gives that $g(f(a)) = g(b) = 0$. But $gf = \mathrm{id}_A$, so $g(f(a)) = b$, hence $\ker \phi$ is trivial. Therefore, ϕ is injective.

Consider any $(a, c) \in A \oplus C$. h is surjective by exactness, so there exists b such that $h(b) = c$. Let $b' = f(a) + b - f(g(b))$. Then

$$\begin{aligned}
\phi(b') &= (g(f(a) + b - f(g(b))), h(f(a) + b - f(g(b)))) \\
&= (g(f(a)) + g(b) - g(f(g(b))), h(f(a)) + h(b) - h(f(g(b)))) && g, h \text{ are homomorphisms} \\
&= (a + g(b) - g(b), h(f(a)) + h(b) - h(f(g(b)))) && gf = \mathrm{id}_A \\
&= (a, h(b)) = (a, c) && \operatorname{im} f = \ker h
\end{aligned}$$

ϕ is a bijective homomorphism, so it is an isomorphism, and therefore $B \cong A \oplus C$. ∎

Exercise 49. *If (X, A) is a pointed pair such that there exists a retraction $r : X \to A$, then $\pi_i(X) \cong \pi_i(A) \oplus \pi_i(X, A)$ for all $i \geqslant 2$.*

Proof. Consider the relative homotopy sequence of pairs derived from the Puppe Sequence given by

$$\dots \to \pi_i(A) \to \pi_i(X) \to \pi_i(X, A) \to \pi_{i-1}(A) \to \dots$$

Since there is a retraction $X \to A$, the map $\pi_i(A) \to \pi_i(X)$ is injective for all $i \geqslant 2$. By **Proposition 6**, the short sequence

$$0 \to \pi_i(A) \to \pi_i(X) \to \pi_i(X, A) \to 0$$

is exact for all $i \geqslant 2$. By definition of retract, the retract composed with the inclusion is the identity map on A. But then by functoriality, the composition of the induced maps is the identity on the homotopy groups. That is, $(r \circ i) = \mathrm{id}$ implies $(r \circ i)_* = r_* \circ i_* = \mathrm{id}_*$. Thus, by **Proposition 7**, the short exact sequence splits, and hence $\pi_i(X) \cong \pi_i(A) \oplus \pi_i(X, A)$ for all $i \geqslant 2$. ∎

> **Exercise 50.** *An n-dimensional, n-connected CW complex is contractible.*

Proof. Let X be an n-dimensional, n-connected CW complex. By CW approximation, there exists ΓX such that $\gamma : X \to \Gamma X$ is a weak-homotopy equivalence and ΓX has a unique 0-cell and no q-cells for $0 < q \leqslant n$. By Whitehead's Theorem, this weak-equivalence is a homotopy equivalence. By cellular approximation, γ is homotopy equivalent to a cellular map. But then this map is a homotopy equivalence from X to a point, since X has no q-cells for $q > n$ and each q-cell for $q \leqslant n$ gets mapped to the unique 0-cell of ΓX. Hence X is contractible. ∎

> **Exercise 51.** *A CW complex retracts onto any contractible subcomplex.*

Proof. Let $h_t : A \to A$ be a homotopy such that $h_0(a) = a_0$ for some $a_0 \in A$ and let $h_1(a) = a$. Let $H_0 : X \to A$ be given by $H_0(x) = a_0$. Note $H_0|_A = h_0$. Since any CW pair satisfies the homotopy extension property, there exists an extension $H_t : X \to A$ such that $H_t|_A = h_t$. But then H_1 gives the desired retract, since $H_1 : X \to A$ and $H_1|_A = h_1 = \mathrm{id}_A$. ∎

Lemma 2. *Let Z be a CW approximation of a space X. If there exists a weak homotopy equivalence $X \to Y$ or $Y \to X$, then Z is a CW approximation of Y.*

Proof. Consider first the case where there exists a weak homotopy equivalence $g : X \to Y$. Let $h : Z \to X$ be the given weak homotopy equivalence. Then $g \circ h : Z \to Y$ is a weak homotopy equivalence from Z to Y, so Z is a CW approximation for Y.

In the latter case, there exists a weak homotopy equivalence $f : Y \to X$. Let W be a CW approximation of Y and let $w : W \to Y$ be the corresponding weak homotopy equivalence. By the previous argument using composition, W is a CW approximation of X. But then by **Corollary 4.19** in Hatcher, there is a homotopy equivalence $k : Z \to W$. Hence, $w \circ k : Z \to Y$ is a weak homotopy equivalence, hence Z is a CW approximation for Y. ∎

> **Exercise 52.** *Consider the equivalence relation $\simeq_w$ generated by weak homotopy equivalence: $X \simeq_w Y$ if there are spaces $X = X_1, X_2, \ldots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. $X \simeq_w Y$ iff X and Y have a common CW approximation.*

Proof. ($\Longrightarrow$) Let Z be a CW approximation for X. Then by n-fold application of **Lemma 1**, Z is a CW approximation for Y, and hence they have a common CW approximation.

($\Longleftarrow$) If Z is a CW approximation of X and Y, then we have the sequence $X \leftarrow Z \to Y$ where both arrows are weak homotopy equivalences since Z is a CW approximation. But then this is the criterion for $X \simeq_w Y$. ∎

> **Exercise 53.** *There is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ if $n > k > 0$.*

Proof. Suppose there is a retraction $r : \mathbb{R}P^n \to \mathbb{R}P^k$ for $n > k > 0$. Then there must be a surjection $\pi_k(\mathbb{R}P^n) \to \pi_k(\mathbb{R}P^k)$. If $k > 1$, then $\pi_k(\mathbb{R}P^n) = 0$ and $\pi_k(\mathbb{R}P^k) = \mathbb{Z}$. But this gives a contradiction since there is no surjection $0 \to \mathbb{Z}$. If $k = 1$, then $\pi_k(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_k(\mathbb{R}P^k) = \mathbb{Z}$, and again there is a contradiction since there is no surjection $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. Thus no retraction exists. ∎

Exercise 54. *Given a sequence of CW complexes $K(G_n, n)$, $n = 1, 2, \dots,$ let X_n be the CW complex formed by the product of the first n of these $K(G_n, n)$'s. Via the inclusions $X_{n-1} \subset X_n$ coming from regarding X_{n-1} as the subcomplex of X_n with the n-th coordinate equal to a basepoint 0-cell of $K(G_n, n)$, we can then form the union of all the X_n's, a CW complex X.*

$\pi_n(X) \cong G_n$ for all n.

Proof. The projection map X to a factor is a fiber bundle with canonical sections, and hence we have the long sequence of homotopy groups. But because of the section, each short exact sequence splits to give that $\pi_n(X) \cong G_n \times 0 \times 0 \dots \cong G_n$. ■

Exercise 55. *For a fiber bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is homotopic to a constant map, the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomoprhisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$, this yields isomorphisms*

$$\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$$
$$\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$$

Thus $\pi_7(S^4)$ and $\pi_{15}(S^8)$ have $\mathbb{Z}$ summands.

Proof. Since $F \hookrightarrow E$ is homotopic to a constant map, the induced map $\pi_n(F) \to \pi_n(E)$ is trivial for all n. By exactness, the kernel of the map $\pi_n(E) \to \pi_n(B)$ is trivial, and thus the map is injective. Therefore, the sequence

$$0 \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to 0$$

is exact. To see that the sequence splits, we construct a map $\pi_{i-1}(F) \to \pi_i(B)$.

Since $F \hookrightarrow E$ is homotopic to a constant map, each map $S^{i-1} \to F$ bounds a disk $D^i \to E$. Composing this disk map with the projection gives a map which sends to boundary of the disk to a single point, so it represents a map $S^i \to B$, and gives an element of $\pi_i(B)$. Since this map is a well-defined homomorphism, the short exact sequence splits by and $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. ■

Exercise 56. *If $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.*

Proof. For each point $p \in S^n$, there exists an open neighborhood U such that $U \times S^k \cong \pi^{-1}(U)$. $U \times S^k$ is an open subset of $S^n \times S^k$, and $\pi^{-1}(U)$ is an open subset of S^m, so by **Theorem 2.26** in Hatcher, $m = n + k$. We proceed by cases on n.

If $n = 0$, then $m = k$ by $m = n + k$, and the fiber bundle has the form $S^k \to S^k \to S^0$. The map $S^k \to S^0$ has to be surjective, and thus S^k must be disconnected, so $k = 0$. But then the fiber bundle is $S^0 \to S^0 \to S^0$, which is impossible, since a surjective map $S^0 \to S^0$ must be a bijection as the spaces have the same finite cardinality, and hence cannot have fiber S^0.

If $n = 1$, then the fiber bundle has the form $S^{k-1} \to S^k \to S^1$ by $m = n + k$. The exact sub-sequence $\pi_1(S^k) \to \pi_1(S^1) \to \pi_0(S^{k-1})$ implies $k = 1$ since otherwise there would exist an exact sequence $0 \to \mathbb{Z} \to 0$. So then the fiber bundle is of the form $S^0 \to S^1 \to S^1$, which agrees with $k = n - 1$ and $m = 2n - 1$.

If $n > 1$ then $k < m$ since $m = n + k$, and hence $\pi_k(S^m) = 0$. Therefore any map $S^k \to S^m$ is null-homotopic, and by **Proposition 6**, $\pi_n(S^n) \cong \pi_n(S^m) \oplus \pi_{n-1}(S^k)$. But

$n < m$, so $\pi_n(S^m) = 0$ and hence $\pi_n(S^n) \cong \pi_{n-1}(S^k) \cong \mathbb{Z}$. In particular, $\pi_{n-1}(S^k)$ is non-trivial, so $n-1 \geqslant k$. Additionally, by the exactness of $\pi_{k+1}(S^n) \to \pi_k(S^k) \to \pi_{k-1}(S^m) \cong 0$, $\pi_{k+1}(S^n) \to \pi_k(S^k) \cong \mathbb{Z}$ is surjective, so $\pi_{k+1}(S^n)$ cannot be trivial, and thus $k \geqslant n-1$. Thus, $k = n-1$. Finally, since $m = n+k$, $m = 2n-1$. ∎

Exercise 57. *Let X be the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point. Its homology groups are $H_0(X) \cong \mathbb{Z}, H_1(X) \cong \mathbb{Z}^2$, with higher homology groups being trivial.*

Proof. Since the space is path-connected, we have that $H_0(X) \cong \mathbb{Z}$.

$H_1(X) = \ker(\partial_1)/\operatorname{im}(\partial_2)$. Since all vertices are identified, ∂_1 is the trivial map, and hence its kernel is $\langle a, b, c\rangle$. We still have $\operatorname{im}(\partial_2) = -a + b - c$, and hence $\ker(\partial_1)/\operatorname{im}(\partial_2) = \langle a, b, c\rangle/\langle -a + b - c\rangle \cong \mathbb{Z}^2$.

Since $\operatorname{im}(\partial_3) = \ker(\partial_2) = 0$, we have that $H_2(X) \cong 0$. The higher homology groups vanish since the boundary maps for all higher simplices are trivial. ∎

Exercise 58. *Let X be the Δ complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k-simplex for each $k \leqslant n$. The homology groups of X all vanish except $H_0(X) \cong \mathbb{Z}$ and $H_n(X)$ if n is even.*

Proof. The case $H_0(X)$ follows since the space is path-connected. For $0 < i < n$, we have $\partial_i\theta_i = \sum_{j=0}^{i}(-1)^j\theta|_{\text{i-face}} = \begin{cases}\theta_{i-1} & \text{i even}\\ 0 & \text{i odd}\end{cases}$. Hence, $H_i(X) \cong \begin{cases} C_i/C_i \cong 0 & \text{i even}\\ 0 & \text{i odd}\end{cases}$. Lastly, $H_n(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}) \cong \ker(\partial_n)$. Therefore, $H_n(X) \cong \mathbb{Z}$ if n is even and trivial otherwise. ∎

Exercise 59. *If A is a retract of X, then the map $H_n(A) \to H_n(X)$ induced by the inclusion is injective.*

Proof. Let $x \in \ker i_*$. Then $(r \circ i)_*(x) = (r_* \circ i_*)(x) = 0$. Hence, $\ker i_* \subset \ker(r \circ i)_* = \ker \mathrm{id}_* = 0$, so $\ker i_* = 0$, and hence i_* is injective. ∎

Exercise 60. *A morphism $f : A \to B$ in* **Ch(Ab)** *is an isomorphism if and only if each $f_n : A_n \to B_n$ is an isomorphism of groups.*

Proof. ($\Longrightarrow$) If f is an isomorphism, then $f|_{A_n}$ is an isomorphism onto its image B_n since the restriction of the inverse of f is the inverse of the restriction of f.

($\Longleftarrow$) We have the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_i \arrow[r, "\partial_i"] \arrow[d, "f_i"] & A_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & \dots \\
\dots \arrow[r] & B_i \arrow[r, "\partial'_i"] & B_{i+1} \arrow[r, "\partial'_{i+1}"] & \dots
\end{tikzcd}$$

Let g_i be the inverse of f_i. It suffices to show that the collection of all g_i is a morphism. But this follows since $f_{i+1}\partial_i = \partial'_i f_i$ by definition of morphism, and by composition on the left by g_{i+1} and on the right by g_i, we have $\partial_i g_i = g_{i+1}\partial'_i$, and hence the inverse is a morphism so f is an isomorphism. ∎

Exercise 61. *If A is a complex with all boundary maps equal to the zero map, then $H_n(A) \cong A_n$.*

Proof. We have $H_n(A) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}) \cong A_n/0 \cong A_n$. ■

Exercise 62. *Given two morphisms $f, g : A \to B$ in* **Ch(Ab)**, *let the sum $f+g : A \to B$ be given by $(f+g)_n = f_n + g_n$. H_n : **Ch(Ab)** $\to$ **Ab** is additive: $H_n(f+g) = H_n(f) + H_n(g)$.*

Proof. $H_n(f+g) = \ker((f+g)_n)/\operatorname{im}((f+g)_{n+1}) = \ker(f_n+g_n)/\operatorname{im}(f_{n+1}+g_{n+1}) = H_n(f) + H_n(g)$. ■

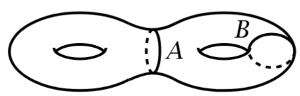
Exercise 63. *A' is a subcomplex of a complex A if each A'_n is a subgroup of A_n and each boundary map on A' is the restriction of the corresponding boundary map on A.*

If $f : A \to B$ is a morphism of complexes, then there is an isomorphism of complexes $A/\ker(f) \cong \operatorname{im}(f)$.

Proof. By the first isomorphism theorem on groups, we have $A_n/\ker(f_n) \cong \operatorname{im}(f_n)$. But the isomoprhism is the quotient map of f, so $\partial g = g\partial$, and hence by **Proposition 4**, we have that there is an isomorphism in **Ch(Ab)** of $A/\ker(f) \cong \operatorname{im}(f)$. ■

Exercise 64. *(a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.*

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed, orientable surface of genus two with A and B the circles shown. [What are X/A and X/B.]



Proof. (a) A is a finite subset of a metric space, hence closed in particular. If A is empty, then the relative homology groups $H_n(X, A)$ reduce to the homology groups $H_n(X)$, which have already been computed in Hatcher. Otherwise, A is nonempty with cardinality $k \in \mathbb{N}$. Let U be the open set given by unions of disjoint open balls centered at each point in A. Then U deformation retracts onto A by contracting each ball to its center. Thus (X, A) is a good pair.

A has k-path components, so $H_0(A) = \mathbb{Z}^k$. Furthermore, A contains no maps of higher dimensional simplices, hence $H_i(A) \cong 0$ for $i > 0$. X is path-connected in both cases, so X/A is path-connected. Thus, $H_0(X, A) \cong \widetilde{H}_0(X/A) \cong 0$.

Consider the long exact sequence

$$\ldots \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0$$

If $X = S^2$, then $H_1(S^2) \cong 0$, so we have the short exact sequence

$$0 \to H_1(X, A) \to \mathbb{Z}^k \to \mathbb{Z} \to 0$$

Since the map $A \to X$ is the inclusion, the induced map has a left inverse, and the sequence splits, so $H_1(X, A) \cong \mathbb{Z}^{k-1}$.

If $X = S^1 \times S^1$, then we have the exact sequence

$$0 \to \mathbb{Z}^2 \to H_1(X, A) \xrightarrow{\partial} \mathbb{Z}^k \to \mathbb{Z} \to 0$$

By exactness, we have that $\ker \partial \cong \mathbb{Z}^2$ and $\operatorname{im} \partial \cong \mathbb{Z}^{k-1}$. Since $H_1(X, A)$ is a finitely generated abelian group with no torsion, it is determined by rank alone. Since its rank has to be $k+1$, we have that $H_1(X, A) \cong \mathbb{Z}^{k+1}$.

In both cases, we have that $H_2(X, A) \cong \mathbb{Z}$.

(b) X/A and X/B are both path-connected, so $H_0(X, A) \cong H_0(X, B) \cong \widetilde{H}_0(X/A) \cong 0$. $H_1(T^2 \vee T^2) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 \cong \mathbb{Z}^4$ by **Corollary 2.25** in Hatcher. X/A is homotopy equivalent to the wedge product $T^2 \vee T^2$, so $H_1(X, A) \cong \widetilde{H}_1(X/A) \cong \widetilde{H}_1(T^2 \vee T^2) \cong \mathbb{Z}^4$. Similarly, $H_2(X, A) \cong \mathbb{Z}^2$. X/B is homotopy equivalent to the wedge product of a torus and a circle, so $H_1(X, B) \cong \widetilde{H}_1(X/B) \cong \widetilde{H}_1(T^2 \vee S^1) \cong \mathbb{Z}^2 \oplus \mathbb{Z} \cong \mathbb{Z}^3$. Similarly, $H_2(X, B) \cong \mathbb{Z}$. ∎

> **Exercise 65.** *For the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.*

Proof. Consider the exact sequence

$$H_1(\mathbb{R}) \to H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial} H_0(\mathbb{Q}) \to H_0(\mathbb{R})$$

$\mathbb{R}$ is contractible, so $H_1(\mathbb{R})$ vanishes and $H_0(\mathbb{R}) \cong \mathbb{Z}$. This gives

$$0 \to H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial} H_0(\mathbb{Q}) \to \mathbb{Z}$$

By exactness, $H_1(\mathbb{R}, \mathbb{Q}) \cong \ker \partial$. $H_0(\mathbb{Q})$ is an infinite product of the integers, since $\mathbb{Q}$ has a path-component for each point in $\mathbb{Q}$. Hence

$$\partial : \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}_q \to \mathbb{Z}$$

where each $\mathbb{Z}_q$ is a copy of $\mathbb{Z}$. But the kernel of this map, for some basepoint $q_0 \in \mathbb{Q}$, has basis $e_q - e_{q_0}$. Hence, since a subgroup of a free abelian group is free abelian, we have that $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian. ∎

> **Exercise 66.** *$\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.*

Proof. Let p and q denote the tips of the two cones that compose SX. Let $U = SX/p$ and $V = SX/q$. By Mayer-Vietoris, we have

$$\ldots \to \widetilde{H}_{n+1}(U) \oplus \widetilde{H}_{n+1}(V) \to \widetilde{H}_{n+1}(SX) \to \widetilde{H}_n(U \cap V) \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \ldots$$

$U \cap V = X \times (0, 1)$, which deformation retracts onto X. Additionally, both U and V deformation retract onto a point, thus are contractible. Hence the Mayer-Vietoris sequence gives the exact sequence

$$0 \to \widetilde{H}_{n+1}(SX) \to \widetilde{H}_n(X) \to 0$$

Hence the two groups are isomorphic for all n.

The second part of the proposition follows by induction. The induction hypothesis is

$$\widetilde{H}_{n+1}\left(\bigcup_{i=1}^{k} CX\right) \cong \bigoplus_{i=1}^{k-1} \widetilde{H}_{n+1}(SX) \cong \bigoplus_{i=1}^{k-1} \widetilde{H}_n(X)$$

The base case is given by the previous part, which also gives the second isomorphism of the inductive hypothesis. By the unnumbered example directly above **Example 2.23** in Hatcher, $CX/X \cong SX$. The first isomorphism in the inductive hypothesis then follows since

$$\widetilde{H}_{n+1}\left(\bigcup_{i=1}^{k} CX\right) \cong \widetilde{H}_{n+1}\left(\bigcup_{i=1}^{k-1} CX, X\right) \cong \widetilde{H}_{n+1}\left(\bigvee_{i=1}^{k-1} SX\right) \cong \bigoplus_{i=1}^{k-1} \widetilde{H}_n(X)$$

■

> **Exercise 67.** *Let $f:(X,A)\to(Y,B)$ be a map such that both $f:X\to Y$ and the restriction $f:A\to B$ are homotopy equivalences.*
> *(a) $f_*:H_n(X,A)\to H_n(Y,B)$ is an isomorphism for all n.*
> *(b) For the case of the inclusion $f:(D^n,S^{n-1})\hookrightarrow(D^n,D^n-\{0\})$, f is not a homotopy equivalence of pairs. That is, there is no $g:(D^n,D^n-\{0\})\to(D^n,S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs. [Observe that a homotopy equivalence of pairs $(X,A)\to(Y,B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures .]*

Proof. (a) The exact sequence of pairs coupled with the homotopy equivalences gives that

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \dots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\
\dots & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(Y) & \longrightarrow & \dots
\end{array}$$

But then the proposition for $n>0$ follows by the five-lemma. For $n=0$, this follows directly by the homotopy equivalences, as homotopy equivalent spaces have the same number of path components.

(b) The observation in the problem statement follows since $f(\overline{A})\supset\overline{f(A)}$. Suppose that the inclusion is a homotopy equivalence. But then by the observation in the problem statement and the previous part, this gives that there is a homotopy equivalence between $\overline{S^{n-1}}=S^{n-1}$ and $\overline{D^n-\{0\}}=D^n$, a contradiction.

■

Lemma 3. *Chain homotopy of chain maps is an equivalence relation.*

Proof. Let $f_n:C_n\to D_n$ be a chain map. Let $\psi_n:C_n\to D_{n+1}$ be given by $\psi_n(\sigma)=0$. Then $\partial_n^D\circ\psi_n+\psi_{n-1}\circ\partial_{n-1}^C=0=f_n-f_n$. Therefore, chain homotopy is reflexive.

Let $f_n,g_n:C_n\to D_n$ be two chain maps and let $\psi_n:C_n\to D_{n+1}$ be a chain homotopy $f\to g$ such that $f_n-g_n=\partial_n^D\circ\psi_n+\psi_{n-1}\circ\partial_{n-1}^C$. Then $-\psi$ is a chain homotopy $g\to f$ since $g_n-f_n=-\partial_n^D\circ\psi_n-\psi_{n-1}\circ\partial_{n-1}^C=\partial_n^D\circ(-\psi_n)+(-\psi_{n-1})\circ\partial_{n-1}^C$. Therefore, chain homotopy is symmetric.

Let $f_n,g_n,h_n:C_n\to D_n$ be three chain maps and let $\psi_n:C_n\to D_{n+1}$ be a chain homotopy $f\to g$ such that

$$f_n-g_n=\partial_n^D\circ\psi_n+\psi_{n-1}\circ\partial_{n-1}^C$$

and let $\varphi_n:C_n\to D_{n+1}$ be a chain homotopy $g\to h$ such that

$$g_n-h_n=\partial_n^D\circ\varphi_n+\varphi_{n-1}\circ\partial_{n-1}^C.$$

Then adding the two previous chain homotopies gives a chain homotopy $f \to h$ since

$$\begin{aligned} f_n - h_n = f_n - g_n + g_n - h_n &= \partial_n^D \circ \psi_n + \psi_{n-1} \circ \partial_{n-1}^C + \partial_n^D \circ \varphi_n + \varphi_{n-1} \circ \partial_{n-1}^C \\ &= \partial_n^D \circ (\psi_n + \varphi_n) + (\psi_{n-1} + \varphi_{n-1}) \circ \partial_{n-1}^C \end{aligned}$$

Therefore chain homotopy is transitive in addition to being reflexive and symmetric, and is thus an equivalence relation. ∎

Lemma 4. *Chain homotopy is compatible with the composition of chain maps.*

Proof. ψ be a chain homotopy $f_1 \to f_2$ and let φ be a chain homotopy $g_1 \to g_2$. Suppressing indices and composition for brevity, we have

$$\begin{aligned} g_1 f_1 - g_2 f_2 &= g_1 f_1 + g_1 f_2 - g_1 f_2 - g_2 f_2 \\ &= g_1(f_1 - f_2) + (g_1 - g_2) f_2 \\ &= g_1(\partial\psi + \psi\partial) + (\partial\varphi + \varphi\partial) f_2 \\ &= \partial g_1 \psi + g_1 \psi \partial + \partial \varphi f_2 + \varphi f_2 \partial \\ &= \partial(g_1 \psi + \varphi f_2) + (g_1 \psi + \varphi f_2)\partial \end{aligned}$$

Therefore $g_1 f_1$ and $g_2 f_2$ are chain homotopic. ∎

> **Exercise 68.** *Let* $\mathbf{K(Ab)}$ *be* $\mathbf{Ch(Ab)}$ *with homotopy classes of maps as its morphisms. More precisely, let ob*$(\mathbf{K(Ab)})$ *be chain complexes in the category of abelian groups. Let hom*$(\mathbf{K(Ab)})$ *be chain maps up to homotopy. That is, two chain maps* $f, g : C \to D$ *are equivalent if and only if there exists a sequence of morphisms* $\psi_n : C_n \to D_{n+1}$ *such that* $f_n - g_n = \partial^D \circ \psi_n + \psi_{n-1} \circ \partial^C$*.* $\mathbf{K(Ab)}$ *is a well-defined category.*

Proof. This follows by the previous two lemma. ∎

> **Exercise 69.** *A complex of abelian groups is called acyclic if its homology groups all vanish. (Similarly, a topological space is called acyclic if its associated singular chain complex is acyclic; i.e., if its singular homology groups all vanish.) A complex of abelian groups A is called contractible if the identity map on A is chain homotopic to the zero map. Prove that all contractible complexes are acyclic, and give an example of an acyclic complex that is not contractible.*

Proof. Let A be a contractible chain complex. Then there exist maps $\psi_n : A_n \to A_{n+1}$ such that

$$\sigma = \partial_{n+1}\psi_{n+1}(\sigma) + \psi_n \partial_n(\sigma).$$

We claim A is acyclic. Since $\operatorname{im}\partial_{n+1} \subset \ker \partial_n$, it suffices to show $\ker \partial_n \subset \operatorname{im}\partial_{n+1}$. If $\sigma \in \ker \partial_n$, then $\sigma = \partial_{n+1}\psi_{n+1}(\sigma) + \psi_n\partial_n(\sigma) = \partial_{n+1}\psi_{n+1}(\sigma) + \psi_n(0) = \partial_{n+1}\psi_{n+1}(\sigma)$, so $\sigma \in \operatorname{im}\partial_{n+1}$.

The chain complex

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is acyclic but not contractible, since if it were contractible, the short exact sequence would split, and this is impossible since $\mathbb{Z}$ is isomorphic to the direct sum of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. ∎

Exercise 70. *$S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covers do not.*

Proof. Since both spaces are path-connected, their homology groups match in dimension zero. $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$ and $H_1(S^1 \vee S^1 \vee S^2) \cong H_1(S^1) \oplus H_1(S^1) \oplus H_1(S^2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \cong \mathbb{Z}^2$, so their homology groups match in dimension one. $H_2(S^1 \times S^1) \cong \mathbb{Z}$ and $H_2(S^1 \vee S^1 \vee S^2) \cong H_2(S^1) \oplus H_2(S^1) \oplus H_2(S^2) \cong 0 \oplus 0 \oplus \mathbb{Z} \cong \mathbb{Z}$, so their homology groups match in dimension two. In dimensions greater than two, the torus has trivial homology groups, and since S^1 and S^2 also have trivial homology groups in such dimensions and the homology group of the wedge product splits up as a direct sum, the homology groups of both spaces match in all dimensions.

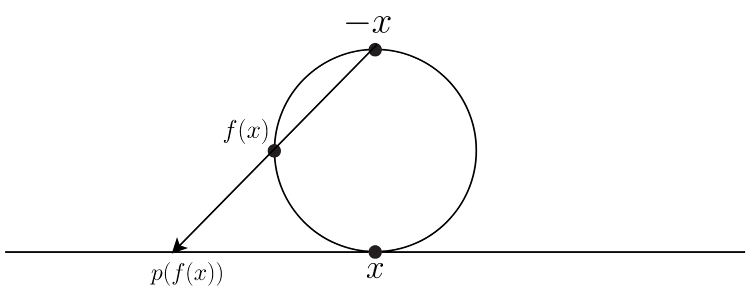
The universal cover of the torus is the plane, which is contractible. Therefore, it suffices to show that the universal cover of $\widetilde{X}$ of $S^1 \vee S^1 \vee S^2$ has a nontrivial homology group of dimension greater than zero. Since $\widetilde{X}$ is simply-connected, $H_2(\widetilde{X}) \cong \pi_2(\widetilde{X})$ by the Hurewicz Theorem. Since $\pi_2(\widetilde{X}) \cong \pi_2(S^1 \vee S^1 \vee S^2) \ncong 0$, we are done. ∎

Exercise 71. *Given a map $f : S^{2n} \to S^{2n}$, there exists $x \in S^{2n}$ such that $f(x) = \pm x$. Therefore, every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. The map $h : S^{2n-1} \to S^{2n-1}$ given by*

$$h(x_1, x_2, \ldots, x_{2n}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2n}, -x_{2n-1})$$

induces a map on $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ with no fixed points.

Proof. Suppose there does not exist x such that $f(x) = -x$. Then define a tangent vector field v on S^{2n} as follows. For each point $x \in S^{2n}$, let $p(f(x))$ denote the stereographic projection of $f(x)$ using $-x$ as the projection point. This is well defined since $f(x) \neq -x$ for all $x \in S^{2n}$. Let $v(x) = p(f(x)) - x$.



v is a continuous vector field since f and the stereographic projection are continuous, and since $2n$ is even, there must exist a point x_0 such that v vanishes by **Theorem 2.28** in Hatcher. This implies $p(f(x_0)) = x_0$. But $p(f(x_0)) = p(x_0)$, so by injectivity, $f(x_0) = x_0$.

Let $f : \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ be any map. Extend f to a map $g : S^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ by $g(x) = g(-x) = f(q(x))$, where q is the natural projection $S^{2n} \to \mathbb{R}\mathrm{P}^{2n}$. By **Proposition 1.33** in Hatcher, there exists a lift $\widetilde{g} : S^{2n} \to S^{2n}$. By the argument above, there exists x such that $\widetilde{g}(x_0) = \pm x_0$. But then $q(\widetilde{g}(x_0)) = q(x_0)$, and since $q \circ \widetilde{g} = g$, we have that $g(x_0) = q(x_0)$. Since $g(x_0) = f(q(x_0))$, $q(x_0)$ is a fixed point of f.

Let h be defined as in the proposition. h descends to a map $\mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ since $q(h(x)) = q(h(-x))$. But since $h(x) \neq \pm x$ for any x, the descent map has no fixed points. ∎

Exercise 72. (a) *If $f : S^n \to S^n$ is a map of degree zero, then there exist $x, y \in S^n$ such that $f(x) = x, f(y) = -y$.*

(b) *If F is a continuous, non-vanishing vector field on D^n, then there exist $x \in \partial D^n$ where F points radially outward and $y \in \partial D^n$ where F points radially inward.*

Proof. (a) By property **g** of degree in Hatcher, since $\deg f \neq (-1)^{n+1}$, it must have a fixed point. Following the proof of property **g** of degree, if $f(x) \neq -x$ for any x, then the line segment $f(x) \to x$ does not pass through the origin. Then define the homotopy

$$f_t(x) = \frac{(1-t)f(x) + tx}{\|(1-t)f(x) + tx\|}$$

from f to the identity map. Since the identity map does not have degree zero, this gives the desired contradiction.

(b) Define $G : D^n \to S^{n-1}$ by $G(x) = F(x)/\|F(x)\|$. The restriction $G|_{\partial D^n} : S^{n-1} \to S^{n-1}$ is equivalent to $G \circ i$ where i is the natural inclusion, so $\deg G|_{\partial D^n} = 0$ as D^n is contractible. The claim follows by the previous argument. ∎

Exercise 73. (a) *The long exact sequence of homology groups associated to the short exact sequence of chain complexes $0 \to C_i(X) \xrightarrow{n} C_i(X) \to C_i(X; \mathbb{Z}/n\mathbb{Z})$ yields short exact sequences*

$$0 \to H_i(X)/nH_i(X) \to H_i(X; \mathbb{Z}/n\mathbb{Z}) \to n\text{-}Torsion(H_{i-1}(X)) \to 0$$

where n-Torsion(G) is the kernel of the map $G \xrightarrow{n} G, g \mapsto ng$.

(b) *$\widetilde{H}_i(X; \mathbb{Z}/p\mathbb{Z}) = 0$ for all i and all primes p if and only if $\widetilde{H}_i(X)$ is a vector space over $\mathbb{Q}$ for all i.*

Proof. (a) We have the long exact sequence

$$\ldots \to H_i(X) \xrightarrow{\eta} H_i(X) \xrightarrow{f_n} H_i(X; \mathbb{Z}/n\mathbb{Z}) \to \ldots$$

Since $\operatorname{im} \eta = \ker f$, we have that $\ker f = nH_i(X)$. By the first isomorphism theorem, $H_i(X)/\ker f \cong H_i(X; \mathbb{Z}/n\mathbb{Z})$. Additionally, $\operatorname{im} \partial = \ker f_{n-1} = \text{n-Torsion}(G)$, we have the sequence

$$0 \to H_i(X)/nH_i(X) \to H_i(X; \mathbb{Z}/n\mathbb{Z}) \to \text{n-Torsion}(H_{i-1}(X)) \to 0$$

which is exact since the first map is injective and since $\partial f = 0$.

(b) ($\implies$) It suffices to show that $\widetilde{H}_i(X)$ is free, abelian, torsion-free, and infinitely-generated.

($\impliedby$) $\mathbb{Q}$ has no torsion and $H_i(X; \mathbb{Q})/nH_i(X; \mathbb{Q}) = 0$, so by the splitting lemma, the claim follows. ∎

Exercise 74. *$H_i(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for all $i \in \mathbb{N}$.*

Proof. There exists a two-sheeted cover $p : S^\infty \to \mathbb{R}\mathrm{P}^\infty$, so by the proof of proposition **2B.6** in Hatcher, the (transfer) sequence

$$\ldots \to H_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_i(S^\infty, \mathbb{Z}/2\mathbb{Z}) \to H_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_{i-1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to \ldots$$

is exact. Since S^∞ is contractible, the sequence is

$$\ldots \to H_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to 0 \to H_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_{i-1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to 0 \to \ldots$$

In particular, we have short exact sequences

$$0 \to H_{i+1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to 0$$

for all $i > 0$. Therefore $H_{i+1}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong H_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$ for all $i > 0$.

We finish the remaining two cases. $\mathbb{R}\mathrm{P}^\infty$ is path-connected, so $H_0(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Adding cells of dimension greater than two does not affect the first homology group, so $H_1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. ∎

Exercise 75. *Let $f : C \to D$ be a morphism of chain complexes. Define C_f as the mapping cone of f such that $(C_f)_n = C_{n-1} \oplus D_n$, with boundary maps $d(x, y) = (-d^C_{n-1}(x), f_{n-1}(x) + d^D_n(y))$.*

C_f is a chain complex.

Proof. We check that $d^{C_f}_n \circ d^{C_f}_{n+1} = 0$. We have

$$d^{C_f}_n = \begin{pmatrix} -d^C_{n-1} & 0 \\ f_{n-1} & d^D_n \end{pmatrix}$$

$$d^{C_f}_n \circ d^{C_f}_{n+1} = \begin{pmatrix} d^C_{n-1} \circ d^C_n & 0 \\ -f_{n-1} \circ d^C_n + d^D_n \circ f_n & d^D_n \circ d^D_{n+1} \end{pmatrix}$$

The diagonal terms are zero by assumption. The last entry is zero by the commutativity of

$$\begin{tikzcd} \dots \arrow[r] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r] & \dots \end{tikzcd}.$$

∎

Exercise 76. *Define a map $j : D \to C_f$ by $j_n(y) = (0, y)$, and a map $d : C_f \to C[-1]$ by $d_n(x, y) = (-1)^n x$. Here $C[-1]$ denotes the same complex as C, but re-indexed so that $C[-1]_n = C_{n-1}$ and $\partial^{C[-1]} = \partial^C$.*

$$0 \to D \xrightarrow{j} C_f \xrightarrow{d} C[-1] \to 0$$

is an exact sequence of complexes.

Proof. If $y \in \ker j_n$, then $y = 0$. Hence, $\ker j = 0$. If $(x, y) \in \ker d$, then $x = 0$. Hence, $\operatorname{im} j = \ker d$. d is surjective, and thus the sequence is exact. ∎

Exercise 77. *A map $f : C \to D$ of complexes is a quasi-isomorphism if it induces isomorphisms on all homology groups. A morphism f is a quasi-isomorphism if and only if C_f is acyclic.*

Proof. ($\implies$) Let $[(x, y)] \in H_n(C_f)$. If (x, y) is a cycle, then $d(x) = 0$ and $d(y) + f(x) = 0$. $f(x) = d(-y)$, so $f(x)$ is a boundary, and by injectivity, x is also a boundary, so $x = d(x')$ and $[x] = 0$. This gives $d(y + f(x')) = 0$, so $y + f(x')$ is a cycle. By surjectivity, $[y + f(x')] = f([x''])$, so $y = f(x'' - x')$. Thus

$$d(x, y) = (-d(x'' - x'), f(x'' - x')) = (x'' - x', 0)$$

so (x, y) is a boundary. Since every cycle is a boundary, C_f is acyclic as all of its homology groups vanish.

($\Longleftarrow$) Suppose $f_*[x] = 0$. Since $[-x, y]$ is a cycle, it is also a boundary as C_f is acyclic. In particular x is a boundary, and hence f_* is injective since its kernel is trivial. Now suppose that $[y]$ is a cycle in D_n. Then $(0, y)$ is a cycle, and hence a boundary as C_f is acyclic. Hence $[y] = [d(y') + +f(x')]$ and hence in the image of f_*, so f is a quasi-isomorphism.

■

Exercise 78. *The homology groups of the following 2-complexes are described in the following proof.*

(*a*) *The quotient of S^2 obtained by identifying north and south poles to a point.*

(*b*) $S^1 \times (S^1 \vee S^1)$.

(*c*) *The space obtained from D^2 by first deleting the interiors of two disjoint sub-disks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.*

(*d*) *The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by a $2\pi/m$ - rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by a $2\pi/n$ - rotation.*

Proof. (*a*) By **Example 0.8** in Hatcher, $X \simeq S^2 \vee S^1$. By **Corollary 2.25** in Hatcher,

$$H_n(X) \cong H_n(S^2 \vee S^1) = \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & n > 2 \end{cases}.$$

(*b*) Let S^1 and $S^1 \vee S^1$ have the CW structures

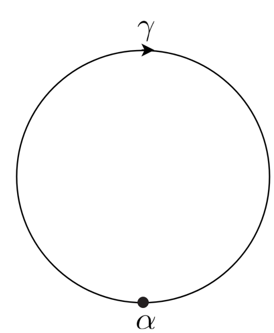


Figure 2. CW complex structure on S^1

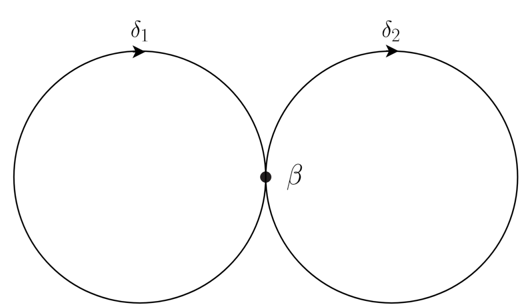


Figure 3. CW complex structure on $S^1 \vee S^1$

By **Theorem A.6** in Hatcher, the cellular chain complex of the product is

$$0 \to \mathbb{Z}\{\gamma \times \delta_1, \gamma \times \delta_2\} \xrightarrow{d=0} \mathbb{Z}\{\alpha \times \delta_1, \alpha \times \delta_2, \beta \times \gamma\} \xrightarrow{d=0} \mathbb{Z}\{\alpha \times \beta\} \to 0$$

Therefore,

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^3 & n = 1 \\ \mathbb{Z}^2 & n = 2 \\ 0 & n > 2 \end{cases}$$

(c) Let X have the following CW complex structure.

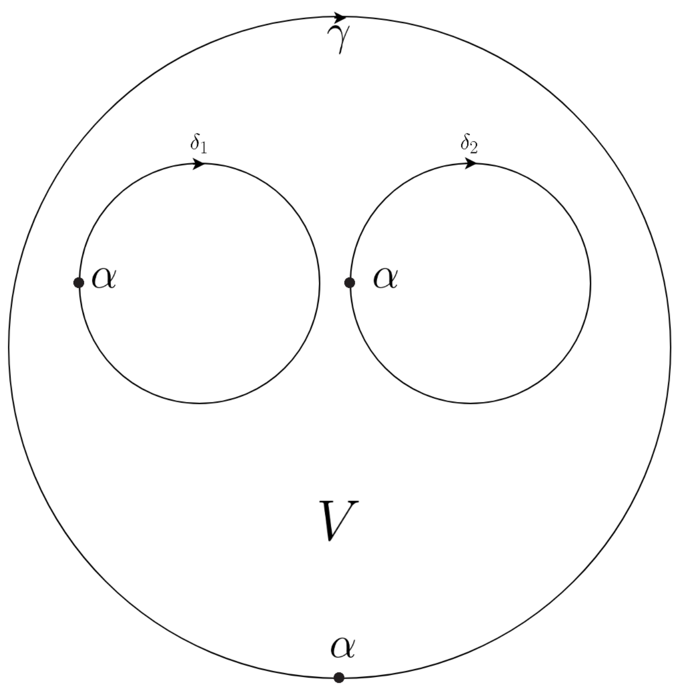


FIGURE 4. CW complex structure on X

Then the cellular chain complex of X is

$$0 \to \mathbb{Z}\{V\} \xrightarrow{d_2} \mathbb{Z}\{\gamma, \delta_1, \delta_2\} \xrightarrow{0} \mathbb{Z}\{\alpha\} \to 0$$

where $d_2(V) = -\gamma$. Hence,

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ 0 & n > 1 \end{cases}$$

(d) We modify the attachment of the two-cell of the usual CW complex structure of the torus to maintain the identification. Given the zero-cell α, and the two one-cells δ_1, δ_2, attach the two-cell along $\delta_1^n \delta_2^m \delta_1^{-n} \delta_2^{-m}$. Thus the cellular chain complex for X is

$$0 \to \mathbb{Z}\{V\} \xrightarrow{0} \mathbb{Z}\{\delta_1, \delta_2\} \xrightarrow{0} \mathbb{Z}\{\alpha\} \to 0$$

Hence,

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

■

Exercise 79. *If X is a CW complex, then $H_n(X^n)$ is free.*

Proof. Consider the cellular map $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. Each $H_n(X^n, X^{n-1})$ is free with generators the n-cells of X, so it suffices to show that $\ker d_n \cong H_n(X^n)$, as the kernel of a homomorphism is a subgroup, and a subgroup of a free group is free. By exactness of the cellular chain complex, $\operatorname{im} d_{n+1} = \ker d_n$. Additionally, by **Hatcher p. 139**, the diagram

$$\begin{tikzcd} & H_n(X^n) \arrow[dr, "j_n"] & \\ H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \end{tikzcd}$$

commutes. Hence, $H_n(X^n)$ is a subgroup of $H_n(X^n, X^{n-1})$, so is free. ■

Exercise 80. *Suppose the space X is the union of open sets $A_1, \ldots, A_n$ such that each inter-section $A_{i_1} \cap \ldots \cap A_{i_k}$ is either empty or has trivial reduced homology groups. Then $\widetilde{H}_i(X) = 0$ for $i \geqslant n-1$, and give an example showing this inequality is best possible, for each n.*

Proof. Let $X_k = \bigcup_{i=1}^k A_i$ and $Y_k = \bigcap_{i=1}^k A_i$. We claim for all k in $\{1, 2, \ldots, n\}$, $\widetilde{H}_i(X_k \cap Y_{k+1})$ vanishes for all $i > k-2$. The proposition follows from the case $k = n$.

The case $k = 1$ follows by assumption. By induction, we have that $\widetilde{H}_i(X_{k-1} \cap Y_{k+1}) = 0$ for all $i > k-3$. This gives the sequence

$$\widetilde{H}_i(X_{k-1} \cap Y_k) \to \widetilde{H}_i(X_{k-1} \cap Y_{k+1}) \oplus \widetilde{H}_i(Y_k) \to \widetilde{H}_i(X_{k-1} \cap Y_{k+1}) \to \widetilde{H}_{i-1}(X_{k-1} \cap Y_k)$$

by Mayer-Vietoris. This yields the exact sequence

$$\widetilde{H}_i(X_{k-1} \cap Y_{k+1}) \to \widetilde{H}_i(X_k \cap Y_{k+1}) \to \widetilde{H}_i(X_{k-1} \cap Y_k)$$

Since the outer terms are zero by induction for all $i > k-2$, the proposition follows.

Consider any $n > 3$ and let $X = S^{n-2}$. Decompose S^{n-2} into n open sets such that the conditions of the proposition hold. Since $\widetilde{H}_{n-2}(X) = \mathbb{Z}$, the proposition gives the best possible bound. ■

Exercise 81. *Let F be a free abelian group, and consider the solid diagram*

$$\begin{tikzcd} & F \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "g"'] & B \end{tikzcd}$$

Then there exists a map h making the diagram commute.

Proof. Let $\{g_i\}$ be a basis of F. Since g is surjective, for all $f(g_i)$, there exists a_i such that $g(a_i) = f(g_i)$. Let $h(g_i) = a_i$. By the universal property of free groups, h extends uniquely to a homomorphism $F \to A$. But then since $g \circ h$ and f agree on where they map generators, they must agree on all of F, again by the universal property and uniqueness. ■

> **Exercise 82.** *If $f : A \to F$ is a surjective map of abelian groups with F free, then $A = \ker f \oplus F'$, where $F' \cong F$.*

Proof. The short sequence

$$0 \to \ker f \xrightarrow{i} A \xrightarrow{f} F \to 0$$

is exact since the inclusion i is injective and f is surjective. Now consider the solid diagram

$$\begin{tikzcd} & F \arrow[dl, dashed, "g"'] \arrow[d, "\mathrm{id}"] \\ A \arrow[r, two heads, "f"'] & F \end{tikzcd}$$

By the previous proposition, there exists a homomorphism g such that $f \circ g = \mathrm{id}$. But then the short exact sequence splits, and $A \cong \ker \oplus F$, so the proposition follows. ■

> **Exercise 83.** *Let (C, d) be a chain complex of abelian groups such that each C_n is free. Then C is quasi-isomorphic to the chain complex H with $H_n = H_n(C)$ and all differentials the zero map. Equivalently, C is formal or quasi-isomorphic to its homology.*

Proof. Consider the solid diagram

$$\begin{tikzcd} \dots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, dashed, "\phi_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, dashed, "\phi_n"] & C_{n-1} \arrow[r] \arrow[d, dashed, "\phi_{n-1}"] & \dots \\ \dots \arrow[r] & \frac{\ker d_{n+1}}{\operatorname{im} d_{n+2}} \arrow[r, "0"] & \frac{\ker d_n}{\operatorname{im} d_{n+1}} \arrow[r, "0"] & \frac{\ker d_{n-1}}{\operatorname{im} d_n} \arrow[r] & \dots \end{tikzcd}$$

We claim there exists a chain map $\phi : C \to H$ such that the diagram commutes and ϕ induces isomorphisms on all homology groups. Explicitly, $\phi_n \circ d_{n+1} = 0$ and $H_n(\phi) : H_n(C) \to H_n(H)$ is an isomorphism for all n.

Consider the short exact sequence

$$0 \to \ker d_n \to C_n \to \operatorname{im} d_n \to 0.$$

By the previous proposition, it splits, and hence $C_n \cong \ker d_n \oplus \operatorname{im} d_n$ by some isomorphism ψ_n. Let $\pi_1 : \ker d_n \oplus \operatorname{im} d_n \to \ker d_n$ be the natural projection onto the first component, and let $q_n : \ker d_n \to \ker d_n / \operatorname{im} d_{n+1}$ be the quotient map to the coset space. Finally, let $\phi_n = q_n \circ \pi_1 \circ \psi_n$.

$\phi_n \circ d_{n+1} = 0$ as $\operatorname{im} d_{n+1} \subset \ker q_n$, so ϕ is a valid chain map. Note that $H_n(H) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}$ since each differential is the zero map. Hence, the induced map

$$H(\phi_n) : \frac{\ker d_n}{\operatorname{im} d_{n+1}} \to \frac{\ker d_n}{\operatorname{im} d_{n+1}}$$

is an isomorphism since ψ_n is an isomorphism.

■

Exercise 84. *A map $f : X \to Y$ between connected, n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for all $i \leqslant n$.*

Proof. Let $p_y : \widetilde{Y} \to Y$ denote the covering map of the universal cover of Y, and let $p_x : \widetilde{X} \to X$ denote the covering map of the universal cover of X. Now consider the map $f \circ p_x : \widetilde{X} \to Y$. Y is a connected CW complex, which is locally contractible, and in particular, path-connected and locally path-connected. Since $\widetilde{X}$ is simply connected, $(f \circ p_x)_*(\pi_1(\widetilde{X})) \cong 0 \subset (p_y)_*(\pi_1(Y))$, and thus by the lifting criterion, the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f \circ p_x}} & \widetilde{Y} \\ {\scriptstyle p_x}\downarrow & & \downarrow{\scriptstyle p_y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes.

p_x and p_y induce isomorphisms on π_i for $i > 1$, and f also induces isomorphisms on π_i for all $i \leqslant n$ by assumption. Therefore, by commutativity of the diagram, $\widetilde{f \circ p_x}$ induces isomorphisms on π_i for $1 < i \leqslant n$. Additionally, since the universal covers are simply connected, $\widetilde{f \circ p_x}$ induces isomorphisms on π_i for $i \leqslant n$.

Since $\widetilde{Y}$ is a deformation retract of $M_{\widetilde{f \circ p_x}}$, we may replace $\widetilde{Y}$ with the mapping cone and regard $\widetilde{f \circ p_x}$ as an inclusion. Since $\widetilde{X}$ and $\widetilde{Y}$ are simply connected, $\pi_1(\widetilde{Y}, \widetilde{X}) = 0$. By the relative Hurewicz theorem, the first nonzero $\pi_i(\widetilde{Y}, \widetilde{X})$ is isomorphic to the first nonzero $H_i(\widetilde{Y}, \widetilde{X})$. By the long exact sequence of homotopy, since $\widetilde{f \circ p_x}$ induces isomorphisms up to π_n, the groups $\pi_i(\widetilde{Y}, \widetilde{X})$ vanish up to $i = n$, and so also then do the groups $H_i(\widetilde{Y}, \widetilde{X})$. Hence there are induced isomorphisms on all homology groups H_i for $i \leqslant n$.

Now the claim that $\widetilde{X}$ and $\widetilde{Y}$ are n-dimensional CW complexes suffices to finish off the proof of the proposition. This is because if the claim is true, then $H_i(\widetilde{Y}, \widetilde{X})$ vanish for $i > n$, and so by **Corollary 4.33** in Hatcher, $\widetilde{f \circ p_x}$ is a homotopy equivalence. But then it must induce isomorphisms on all homotopy groups π_i for all $i > n$, and hence so must f by commutativity of the diagram. The proposition then follows by Whitehead's theorem. The claim itself regarding universal covers of CW complexes is proven in detail in Whitehead, J. H. C. Combinatorial homotopy. I. Bull. Amer. Math. Soc. 55 (1949), 213–245. Each attaching map of the base CW complex can be lifted to the universal cover as cells are contractible, and so there is one cell per each lift corresponding to the base space. This cell decompositon has the right topology since the covering map is a local homeomorphism. ∎

Exercise 85. *Let X be an $(n-1)$-connected CW complex with $n > 1$.*

(*a*) *The natural maps $\pi_{n+1}(X^{n+1}) \to \pi_{n+1}(X)$ and $H_{n+1}(X^{n+1}) \to H_{n+1}(X)$ are surjective.*

(*b*) *The Hurewicz map $\pi_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1})$ is surjective.*

(*c*) *The map $\pi_{n+1}(X) \to H_{n+1}(X)$ is surjective.*

(*d*) *If $n = 1$, the corresponding statement is false.*

Proof. (*a*) Consider an element $\gamma \in \pi_{n+1}(X)$. By cellular approximation, it can be represented by a cellular map $S^{n+1} \to X$. Hence, we can find a representative which factors over the inclusion $X^{n+1} \to X$, thus γ lies in the image of the induced map of the inclusion. The case for homology follows by **Lemma 2.34** in Hatcher.

(b) Consider the commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(X^n) \arrow[r, "a"] \arrow[d, "h_1"] & \pi_{n+1}(X^{n+1}) \arrow[r, "b"] \arrow[d, "h_2"] & \pi_{n+1}(X^{n+1}, X^n) \arrow[r, "c"] \arrow[d, "h_3"] & \pi_n(X^n) \arrow[d, "h_4"] \\
H_{n+1}(X^n) \arrow[r, "a'"] & H_{n+1}(X^{n+1}) \arrow[r, "b'"] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "c'"] & H_n(X^n)
\end{tikzcd}$$

given by the long exact sequence for the pair (X^{n+1}, X^n) associated to the Hurewicz map h. By cellular homology, $H_{n+1}(X^n)$ vanishes, so a' is injective. Furthermore, h_3 and h_4 are isomorphisms by the Hurewicz theorem. The claim follows by a diagram chase. Consider any element y in $\pi_{n+1}(X^{n+1})$. Let $k = h_3^{-1}(b'(y))$. Now $h_4(c(k)) = c'(h_3(k)) = c'(b'(y)) = 0$ by commutativity and exactness. By injectivity of h_4, $c(k)$ is trivial, so by exactness, there exists x such that $b(x) = k$. By commutativity, $h_3(b(x)) = b'(h_2(x)) = b'(y)$. By injectivity of a', $h_2(x) = y$.

(c) The composition of $\pi_{n+1}(X^{n+1}) \to H_{n+1}(X)$ and $\pi_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1})$ is a surjective map, which is a restriction of h, so h must be surjective.

(d) The torus, which is 0-connected, has contractible universal cover, and hence trivial homotopy. But $H_2(\mathbb{T}) \cong \mathbb{Z}$ as shown previously, so no surjection exists.

■

> **Exercise 86.** *The tensor product of two chain complexes $C \otimes C'$ is a chain complex, where the differential map is given by*
> $$\partial^{C\otimes C'}(c, c') = (\partial^C(c), c') + (-1)^{\deg(c)}(c, \partial^{C'}(c')).$$

Proof. Composing twice, we have

$$\begin{aligned}
\partial^2(c \otimes c') &= \partial(\partial^C(c) \otimes c' + (-1)^i c \otimes \partial^{C'}(c')) \\
&= (-1)^{i-1}\partial^C(c) \otimes \partial^{C'}(c') + (-1)^i \partial^C(c) \otimes \partial^{C'}(c') \\
&= 0
\end{aligned}$$

■

> **Exercise 87.** *Let F be a field and X be a space such that $H_i(X; F)$ has finite dimension for all i. Define the Poincaré series p_X to be the formal power series*
> $$p_X(t) = \sum_i (\dim_F H_i(X; F))t^i.$$
> *The proof below gives formulas for the Poincaré series of $S^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{R}P^\infty, \mathbb{C}P^\infty$, and the orientable surface M_g of genus g. If X and Y are spaces with well-defined Poincaré series, then $p_{X \coprod Y}(t) = p_X(t) + p_Y(t)$ and $p_{X \vee Y}(t) = p_X(t) + p_Y(t) - 1$.*

Proof. S^n: $H_i(\mathrm{S}^n; F) = \begin{cases} F & i \in \{0, n\} \\ 0 & \text{else} \end{cases}$. Hence, $p_{S^n}(t) = 1 + t^n$.

$\mathbb{R}\mathrm{P}^n$: We have $p_{\mathbb{R}\mathrm{P}^n}(t) = \begin{cases} 1+t^n & n \text{ odd} \\ 1 & \text{else} \end{cases}$. This follows since for n even, all homology groups of dimension greater than zero with coefficients in a 2-divisible ring vanish, while for n odd, the n-th homology group does not vanish as we have shown previously.

$\mathbb{C}\mathrm{P}^n$: Complex projective space has homology 0 in all odd dimensions and homology $H_i(\mathbb{C}\mathrm{P}^n; \mathbb{Z} = \mathbb{Z}$ in all even dimension up to $2n$. Hence, $p_{\mathbb{C}\mathrm{P}^n}(t) = \sum_{i=0}^{n} t^{2n}$.

$\mathbb{R}\mathrm{P}^\infty$: All homology groups are with coefficients in $\mathbb{Z}/2\mathbb{Z}$ are $\mathbb{Z}/2\mathbb{Z}$, and hence $p_{\mathbb{R}\mathrm{P}^\infty}(t) = \sum_{i=0}^{\infty} t^n$.

$\mathbb{C}\mathrm{P}^\infty$: $p_{\mathbb{R}\mathrm{P}^\infty}(t) = \sum_{i=0}^{\infty} t^{2n}$ by Hatcher's description of the homology being $\mathbb{Z}/2\mathbb{Z}$ in each even dimension.

M_g: By using the identification with the connected sum of tori, we know the homology groups are $\mathbb{Z}$ for $H_0(M;\mathbb{Z})$ and $H_2(M;\mathbb{Z})$. Additionally, $H_1(M;\mathbb{Z}) = \mathbb{Z}^{2g}$. Hence, $p_{M_g}(t) = 1 + 2gt + t^2$.

For $p_{X \coprod Y}(t) = p_X(t) + p_Y(t)$, we know that $H_i(X \coprod Y; F) = H_i(X;F) \oplus H_i(Y;F)$, and thus the dimensions add and the result follows by linearity.

Additionally, $H_i(X \vee Y; F) = \widetilde{H}_i(X \coprod Y; F)$. This gives the same result for $i > 0$, but for $i = 0$, it vanishes, so $p_{X \bigvee Y}(t) = p_X(t) + p_Y(t) - 1$. ■

> **Exercise 88.** *If F is a field, the Künneth formula reduces to a natural isomorphism*
> $$H_n(X \times Y; F) \cong \bigoplus_{p+q=n} H_p(X;F) \otimes_F H_q(Y;F).$$
> *If X and Y are spaces with well-defined Poincaré series, then $p_{X \times Y}(t) = p_X(t) p_Y(t)$.*

Proof. The formula reduces since if F is a field, the groups Tor_F vanish, giving that the map in the short exact sequence in the proposition is an isomorphism. The Poincaré formula follows since

$$\begin{aligned} \dim_F(H_n(X \times Y; F)) &= \sum_{p+q=n} \dim_F(H_p(X;F) \otimes H_q(Y;F)) \\ &= \sum_{p+q=n} \dim_F H_p(X;F) \cdot \dim_F H_q(Y;F) \end{aligned}$$

by the fact that the dimension of a product is the product of the dimensions. But this is exactly the formula for the coefficients of the product of two polynomials, hence $p_{X \times Y}(t) = p_X(t) p_Y(t)$. ■

> **Exercise 89.** *Let X be the CW complex obtained by attaching two 2-cells to S^1, one via a degree p map and one via a degree q map. Below is a description of the homology of X and when X is equivalent to S^2.*

Proof. Taking care of the homology groups independent of p and q, note that $H_n(X) = 0$ for $n > 2$ since X has no n-cells for $n > 2$. Additionally, since X is non-empty and path-connected, $H_0(X) \simeq \mathbb{Z}$.

Now for some notational setup. Give S^1 the standard CW complex structure of one 0-cell e^0 and one 1-cell e^1 with the usual attaching map, and let e^2_p and e^2_q denote the two 2-cells attached via a degree p map and a degree q map, respectively. X is connected and has only one 0-cell, so the differential map $d_1 = 0$ by Hatcher, p. 140. Furthermore, by assumption,

$d_2(e_p^2) = pe^1$ and $d_2(e_q^2) = qe^1$. The cellular chain complex of X is

$$0 \to \mathbb{Z} \oplus \mathbb{Z}\{e_p^2, e_q^2\} \xrightarrow{d_2} \mathbb{Z}\{e^1\} \xrightarrow{d_1} \mathbb{Z}\{e^0\} \to 0.$$

Let's pin-down the kernel and image of d_2. The trivial case is if the degrees of both attaching maps are zero, that is, $p = q = 0$. Then $d_2(me_p^2 + ne_q^2) = 0 + 0 = 0$, so $H_1(X) = \ker(d_1)/\operatorname{im}(d_2) \simeq \mathbb{Z}/0 \simeq \mathbb{Z}$ and $H_2(X) = \ker(d_2) \simeq \mathbb{Z}^2$. In this case, X and S^2 cannot be (weakly) homotopy equivalent since their homology groups are not isomorphic.

Otherwise, suppose either $p \neq 0$ or $q \neq 0$. Any pair $(m, n) \in \mathbb{Z}^2$ maps as $d_2(me_p^2 + ne_q^2) = md_2(e_p^2) + nd_2(e_q^2) = mpe^1 + nqe^1 = (mp + nq)e^1$, using the homomorphism property. By Bézout's Identity, all integers of the form $(mp + nq)$ are multiples of (p, q), the greatest common divisor of p and q. Hence, $\operatorname{im} d_2 \simeq (p, q)\mathbb{Z}$, and $H_1(X) \simeq \mathbb{Z}/(p, q)\mathbb{Z}$. If $(p, q) \neq 1$, then X is not homotopy equivalent to S^2 since $H_1(X) \simeq \mathbb{Z}/(p, q)\mathbb{Z} \not\simeq 0 = H_1(S^2)$.

Now to understand the kernel. By the above, the kernel of d_2 is given by all points $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ such that $pm = -qn$, which is generated by $(q/(p, q), -p/(p, q))$ since $pq/(p, q)$ is the least common multiple of p and q. Hence, $\ker d_2 \simeq \mathbb{Z}$ and thus $H_2(X) = \ker d_2 \simeq \mathbb{Z}$.

If $(p, q) = 1$, we claim that we can construct a map $f : X \to S^2$ which induces an isomorphism on the homology groups, and thus is a homotopy equivalence. It is necessary to note a small, technical detail here. We have only shown 'Whitehead's Theorem for homology', **Corollary 4.33** in Hatcher, for singular homology. Therefore, we need f to induce an isomorphism on singular homology. However, if f induces an isomorphism on cellular homology, and f is a cellular map, then the induced map corresponds naturally to the isomorphism between cellular and singular homology, so a cellular map suffices. This is exercise **2.2.17** in Hatcher, which follows by the naturality of the long exact sequence of homology, the proof of which is not included here for brevity. We proceed by constructing the claimed cellular map.

For one, the induced isomorphism on H_2 must be an isomorphism between the kernels of the differential maps, since no cells of dimension three or greater are attached. The kernel of d_2 is generated by $(q, -p)$ in this case since p and q are relatively prime by the argument above, and the kernel of d_2' is $\mathbb{Z}$ since this differential is trivial for S^2 using the CW complex structure of one 2-cell and one 0-cell. Therefore, the induced map must send $(q, -p) \mapsto 1$.

Then, to create the cellular map which induces the desired map, let m, n be the Bézout coefficients, the two integers such that $pm + qn = 1 = (p, q)$. Let e^2 be the 2-cell of S^2. Then the cellular map is given by mapping $e_p^2 \mapsto e^2$ via any degree n map, and mapping $e_q^2 \mapsto e^2$ via any degree $-m$ map. This gives that $f_*(q, -p) = qn - (-m)p = 1$. f can also map the 0-cell of X to the 0-cell of S^2, and the 1-cell of X to the 0-cell of S^2 or any other point, thus inducing isomorphisms on H_1 and H_0 as well. f thus induces isomorphisms on the cellular homology groups, and since it is cellular, its induced map also induces an isomorphism on the singular homology groups, so it must be a homotopy equivalence by **Corollary 4.33** in Hatcher. ■

Exercise 90. *Any continuous $f : S^n \to S^n$ such that* $\deg f \neq (-1)^{n+1}$ *has a fixed point.*

Proof. This is the contrapositive of property g of degree in Hatcher, p. 134. Below is a reproduction of the proof for completeness.

Suppose that f has no fixed point. Then the line segment $\gamma : I \to \mathbb{R}^{n+1}$ from $f(x)$ to $-x$ given by $\gamma_t(x) = (1 - t)f(x) - tx$ does not pass through the origin. Thus, there exists a homotopy

$$f_t(x) = \frac{\gamma_t(x)}{\|\gamma_t(x)\|}$$

from $f(x)$ to the antipodal map. But by property **f** of degree in Hatcher, p. 134, the antipodal map has degree $(-1)^{n+1}$ since it is the composition of $(n+1)$ reflections in $\mathbb{R}^{n+1}$, each changing the sign of one coordinate. Hence, by property **c** of degree in Hatcher, $\deg f = (-1)^{n+1}$. ∎

Exercise 91. *Recall than an equivalence of categories is a functor $F : C \to C'$, a functor $G : C' \to C$, and two natural isomorphisms $\epsilon : FG \to id_{C'}$ and $\eta : id_C \to GF$. A natural isomorphism is a natural equivalence such that each defining morphism is an isomorphism, and id_C and $id_{C'}$ are the identity functors. Suppose that C has all limits. Then C' has the same property and if $\alpha : D \to C$ is a diagram in C, indexed by a small category D, $\lim(F\alpha) = F(\lim \alpha)$ in the sense that for any choice of the limits, the results are uniquely isomorphic.*

Proof. For all diagrams $\alpha' : D' \to C'$ in C', $G\alpha' : D' \to C$ yields a diagram in C. By assumption, C has a limit (L, ψ) of $G\alpha'$. We claim $F((L,\psi))$ is a limit of α'.

First, we must show it is a cone. For all objects X and Y in D', and all morphisms $f : X \to Y$, the diagram

(1)
$$\begin{tikzcd} & L \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\ (G\alpha')(X) \arrow[rr, "(G\alpha')(f)"'] & & (G\alpha')(Y) \end{tikzcd}$$

commutes by functoriality of $G\alpha'$ and the fact that (L,ψ) is a limit of $G\alpha'$, and in particular, a cone. By functoriality of F, the diagram

(2)
$$\begin{tikzcd} & F(L) \arrow[dl, "F(\psi_X)"'] \arrow[dr, "F(\psi_Y)"] & \\ (FG)(\alpha'(X)) \arrow[rr, "(FG)(\alpha'(f))"'] & & (FG)(\alpha'(Y)) \end{tikzcd}$$

commutes. Now, since ϵ is a natural isomorphism, it is in particular a natural transformation, so the diagram

(3)
$$\begin{tikzcd} (FG)(\alpha'(X)) \arrow[r, "(FG)(\alpha'(f))"] \arrow[d, "\epsilon_{\alpha'(X)}"'] & (FG)(\alpha'(Y)) \arrow[d, "\epsilon_{\alpha'(Y)}"] \\ \mathrm{id}_{C'}(\alpha'(X)) \arrow[r, "\mathrm{id}_{C'}(\alpha'(f))"'] & \mathrm{id}_{C'}(\alpha'(Y)) \end{tikzcd}$$

commutes. The bottom row of 3 can simply be replaced by $\alpha'(X) \xrightarrow{\alpha'(f)} \alpha'(Y)$ by the definition of $\mathrm{id}_{C'}$. Combining 3 and 2 together yields the commutative diagram

$$\begin{tikzcd}
& F(L) \arrow[dl, "F(\psi_X)"'] \arrow[dr, "F(\psi_Y)"] & \\
(FG)(\alpha'(X)) \arrow[rr, "(FG)(\alpha'(f))"] \arrow[d, "\epsilon_{\alpha'(X)}"'] & & (FG)(\alpha'(Y)) \arrow[d, "\epsilon_{\alpha'(Y)}"] \\
\alpha'(X) \arrow[rr, "\alpha'(f)"'] & & \alpha'(Y)
\end{tikzcd} \tag{4}$$

Thus, $F(L)$ is a cone to α'.

Suppose (M, ϕ) is another cone to α'. This yields the commutative diagram

$$\begin{tikzcd}
& M \arrow[ddddl, bend right, "\phi_X"'] \arrow[ddddr, bend left, "\phi_Y"] & \\
& & \\
& F(L) \arrow[dl, "F(\psi_X)"'] \arrow[dr, "F(\psi_Y)"] & \\
(FG)(\alpha'(X)) \arrow[rr, "(FG)(\alpha'(f))"] \arrow[d, "\epsilon_{\alpha'(X)}"'] & & (FG)(\alpha'(Y)) \arrow[d, "\epsilon_{\alpha'(Y)}"] \\
\alpha'(X) \arrow[rr, "\alpha'(f)"'] & & \alpha'(Y)
\end{tikzcd} \tag{5}$$

By functoriality of G applied to the outer morphisms of 5,

$$\begin{tikzcd}
& G(M) \arrow[dl, "G(\phi_X)"'] \arrow[dr, "G(\phi_Y)"] & \\
G(\alpha'(X)) \arrow[rr, "G(\alpha'(f))"'] & & G(\alpha'(Y))
\end{tikzcd} \tag{6}$$

commutes. Now, since (L, ψ) is a limit of $G\alpha'$, there exists a unique map u such that

$$\begin{tikzcd}
& G(M) \arrow[d, "u"] \arrow[ddl, bend right, "G(\phi_X)"'] \arrow[ddr, bend left, "G(\phi_Y)"] & \\
& L \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\
G(\alpha'(X)) \arrow[rr, "G(\alpha'(f))"'] & & G(\alpha'(Y))
\end{tikzcd} \tag{7}$$

commutes. By functoriality of F, diagram 7, and diagram 5, the diagram

$$\begin{tikzcd}
 & & M \arrow[ddddll, bend right=60, "\phi_X"'] \arrow[ddddrr, bend left=60, "\phi_Y"] & & \\
 & & (FG)(M) \arrow[ddll, bend right, "(FG)(\phi_X)"'] \arrow[d, "F(u)"] \arrow[ddrr, bend left, "(FG)(\phi_Y)"] & & \\
 & & F(L) \arrow[dll, "F(\psi_X)"'] \arrow[drr, "F(\psi_Y)"] & & \\
(FG)(\alpha'(X)) \arrow[rrrr, "(FG)(\alpha'(f))"] \arrow[d, "\epsilon_{\alpha'(X)}"'] & & & & (FG)(\alpha'(Y)) \arrow[d, "\epsilon_{\alpha'(Y)}"] \\
\alpha'(X) \arrow[rrrr, "\alpha'(f)"'] & & & & \alpha'(Y)
\end{tikzcd} \tag{8}$$

.

commutes. Flattening out right square of 8 yields the solid commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\phi_Y"] & \alpha'(Y) \\
(FG)(M) \arrow[u, dashed, "\epsilon_M"] \arrow[r, "(FG)(\phi_Y)"] & (FG)(\alpha'(Y)) \arrow[u, "\epsilon_{\alpha'(Y)}"']
\end{tikzcd} \tag{9}$$

where the map ϵ_M exists making the diagram commute since ϵ is a natural transformation $FG \to \mathrm{id}_{C'}$. Additionally, ϵ_M has an inverse ϵ_M^{-1} since ϵ is also a natural isomorphism. By the same argument, we have commuativity of the left square. Thus, comibining these facts, and diagrams 9 and 8 yields the commutative diagram

$$\begin{tikzcd}
 & & M \arrow[ddddll, bend right=60, "\phi_X"'] \arrow[ddddrr, bend left=60, "\phi_Y"] \arrow[d, bend left, "\epsilon_M^{-1}"] & & \\
 & & (FG)(M) \arrow[u, bend left, "\epsilon_M"] \arrow[ddll, bend right, "(FG)(\phi_X)"'] \arrow[d, "F(u)"] \arrow[ddrr, bend left, "(FG)(\phi_Y)"] & & \\
 & & F(L) \arrow[dll, "F(\psi_X)"'] \arrow[drr, "F(\psi_Y)"] & & \\
(FG)(\alpha'(X)) \arrow[rrrr, "(FG)(\alpha'(f))"] \arrow[d, "\epsilon_{\alpha'(X)}"'] & & & & (FG)(\alpha'(Y)) \arrow[d, "\epsilon_{\alpha'(Y)}"] \\
\alpha'(X) \arrow[rrrr, "\alpha'(f)"'] & & & & \alpha'(Y)
\end{tikzcd} \tag{10}$$

.

This yields that the mediating map of the limit, $F(u)\epsilon_M^{-1} : M \to F(L)$ always exists. It remains to show that this map is unique.

Any map $\mu : M \to F(L)$ factors through $(FG)(M)$ using ϵ_M, since

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[dr, "\epsilon_M^{-1}"'] & F(L) \\
& (FG)(M) \arrow[u, "\mu\epsilon_M"']
\end{tikzcd}$$

commutes. Thus, if the map $(FG)(M) \to F(L)$ is unique, then μ is unique. Hence, it suffices to show that the map $F(u)$ is unique in 10. Suppose there exists μ such that

$$\begin{tikzcd}
& (FG)(M) \arrow[d, "\mu"] \arrow[ddl, bend right, "(FG)(\phi_X)"'] \arrow[ddr, bend left, "(FG)(\phi_Y)"] & \\
& F(L) \arrow[dl, "F(\psi_X)"'] \arrow[dr, "F(\psi_Y)"] & \\
(FG)(\alpha'(X)) \arrow[rr, "(FG)(\alpha'(f))"] & & (FG)(\alpha'(Y))
\end{tikzcd} \tag{11}$$

commutes. We aim to show $\mu = F(u)$. By functoriality of G, and application of the natural isomorphism η, and the fact that (L, ψ) is a limit of $G\alpha'$, the diagram

$$\begin{tikzcd}
& & G(M) \arrow[d, bend left, "\eta_M^{-1}"] \arrow[dddddll, bend right=60, "G(\phi_X)"'] \arrow[dddddrr, bend left=60, "G(\phi_Y)"] & & \\
& & (GFG)(M) \arrow[u, bend left, "\eta_M"] \arrow[d, "G(\mu)"] \arrow[ddddll, bend right, "(GFG)(\phi_X)"'] \arrow[ddddrr, bend left, "(GFG)(\phi_Y)"] & & \\
& & (GF)(L) \arrow[d, "\eta_L"] \arrow[ddll, "(GF)(\psi_X)"'] \arrow[ddrr, "(GF)(\psi_Y)"] & & \\
& & L \arrow[dddll, dashed] \arrow[dddrr, dashed] & & \\
(GF)(G(\alpha'(X))) \arrow[rrrr, "(GF)(G(\alpha'(f)))"] \arrow[d, "\eta_{G(\alpha'(X))}"] & & & & (GF)(G(\alpha'(Y))) \arrow[d, "\eta_{G(\alpha'(Y))}"'] \\
G(\alpha'(X)) \arrow[rrrr, "G(\alpha'(f))"'] & & & & G(\alpha'(Y))
\end{tikzcd} \tag{12}$$

commutes. Note that $\eta_L G(\mu)\eta_M^{-1} : G(M) \to L$ is a mediating map for (L, ψ), so by uniqueness of the limit, we must have that $u = \eta_L G(\mu)\eta_M^{-1}$. This equation is just the commutative diagram

$$\begin{tikzcd}
(GFG)(M) \arrow[r, "G(\mu)"] \arrow[d, "\eta_M"'] & (GF)(L) \arrow[d, "\eta_L"] \\
G(M) \arrow[r, "u"] & L
\end{tikzcd} \tag{13}$$

We also have the commutative diagram of the natural transformation

$$\begin{tikzcd}
(GFG)(M) \arrow[r, "(GF)(u)"] \arrow[d, "\eta_M"'] & (GF)(L) \arrow[d, "\eta_L"] \\
G(M) \arrow[r, "u"] & L
\end{tikzcd} \tag{14}$$

By the previous two diagrams, $\eta_L G(\mu) = \eta_L G(F(u))$ so $G(\mu) = G(F(u))$ as η_L is an isomorphism. Composing with F on both sides yields $(FG)(\mu) = (FG)(F(u))$. Now, using the natural transformation ϵ gives the commutative diagrams

$$\begin{tikzcd}
(FGFG)(M) \arrow[r, "(FG)(\mu)"] \arrow[d, "\epsilon_{(FG)(M)}"'] & (FGF)(L) \arrow[d, "\epsilon_{F(L)}"] \\
(FG)(M) \arrow[r, "\mu"] & F(L)
\end{tikzcd}$$

$$\begin{tikzcd}
(FGFG)(M) \arrow[r, "(FGF)(u)"] \arrow[d, "\epsilon_{(FG)(M)}"'] & (FGF)(L) \arrow[d, "\epsilon_{F(L)}"] \\
(FG)(M) \arrow[r, "F(u)"] & F(L)
\end{tikzcd}$$

Writing commutativity down explicitly, $\epsilon_{F(L)}(FG)(\mu) = \mu\epsilon_{(FG)(M)}$ and $\epsilon_{F(L)}(FG)(F(u))) = F(u)\epsilon_{(FG)(M)}$. Since $(FG)(\mu) = (FG)(F(u))$, we have that $F(u)\epsilon_{(FG)(M)} = \mu\epsilon_{(FG)(M)}$, or that $\mu = F(u)$ by composing on the left with $\epsilon_{(FG)(M)}$. Thus the mediating map is unique, and $F((L,\psi))$ is a limit of α', and thus C' has all limits.

We now show that $F(\lim\alpha) = \lim(F\alpha)$. This follows directly from two claims:

(a) If A and B are limits over the same diagram, then there is a unique isomorphism $A \to B$.

(b) $\lim(F\alpha)$ and $F(\lim\alpha)$ are limits over the same diagram.

We have already proven claim (a) in class, using the universal property and uniqueness of the mediating map for limits. For claim (b), $\lim(F\alpha)$ is a limit over $F\alpha$ by definition. We have shown above that given the equivalence of categories, if (L,ψ) is a limit of α, then $F((L,\psi))$ is a limit of $F\alpha$. ∎

> **Exercise 92.** *Let X denote $(S^1)^3$, the three-dimensional torus, with its natural product CW complex structure. Let $f : S^3 \to S^2$ be the Hopf fibration and $g : X \to S^3$ the map collapsing the two-skeleton of X to a point. $f \circ g$ induces the trivial map on all π_n and all $\widetilde{H}_n$, but is not nullhomotopic.*

Proof. Since S^1 is path-connected, $\pi_n((S^1)^3) \simeq \pi_n(S^1) \times \pi_n(S^1) \times \pi_n(S^1)$ by **Proposition 4.2** in Hatcher. Thus, $\pi_n((S^1)^3)$ is trivial for $n \neq 1$ since $\pi_n(S^1) = 0$ for $n \neq 1$, and thus the induced map $(f \circ g)_* : \pi_n((S^1)^3) \to \pi_n(S^2)$ is trivial for all $n \neq 1$. But for $n = 1$, $\pi_1(S^2) = 0$, so the induced map must be trivial for all n. Note that $\widetilde{H}_n(S^3) = 0$ for $n \neq 3$, and $\widetilde{H}_n(S^2) = 0$ for $n \neq 2$, so the induced map $f_* : \widetilde{H}_n(S^3) \to \widetilde{H}_n(S^2)$ must be trivial for all n, and hence the induced map $(f \circ g)_*$ must also be trivial for all n.

Suppose for a contradiction that $f \circ g$ is null-homotopic via a homotopy $h : (S^1)^3 \times I \to S^2$ where $h(x, 0) = (f \circ g)(x)$ and $h(x, 1) = s_0$ for some $s_0 \in S^2$. Since f is a fibration, there exists a lift $\widetilde{h}$ such that

$$\begin{tikzcd}
(S^1)^3 \arrow[r, "g"] \arrow[d, "(S^1)^3 \times \{0\}"'] & S^3 \arrow[d, "f"] \\
(S^1)^3 \times I \arrow[r, "h"'] \arrow[ur, "\widetilde{h}"] & S^2
\end{tikzcd}$$

commutes. By commutativity, $f \circ \widetilde{h} = h$, and since $h(x, 1) = s_0$, it follows that $\widetilde{h}(x, 1)$ is contained in $f^{-1}(s_0) = S^1$. Thus $(\widetilde{h}_1)_*$ factors as a map $(S^1)^3 \to S^1 \to S^3$, but the first map must induce the trivial map on the homology groups of dimension three since $H_3(S^1) = 0$. Hence, $g_* : H_3((S^1)^3) \to H_3(S^3)$ must be trivial, since $g = \widetilde{h}_0$ is homotopic to $\widetilde{h}_1$ via $\widetilde{h}$. But g takes the 3-cell of $(S^1)^3$ to the 3-cell of S^3 via a degree one map, and since it is cellular, the induced a map on the cellular chain complexes cannot be trivial due to its degree as $H_3((S^1)^3) \simeq \mathbb{Z}$ by Hatcher, p. 143, and $H_3(S^3) \simeq \mathbb{Z}$. ■

> **Exercise 93.** *Let $i : A \to X$ be an inclusion. Show that i is null-homotopic if, and only if, X is a retract of the mapping cone C_i of i. If i is nullhomotopic, $H_n(X, A) \simeq \widetilde{H}_n(X) \oplus \widetilde{H}_{n-1}(A)$ for each $n \geqslant 1$.*

Proof. To codify conventions, let C_i be the space $A \times I \coprod X$ with the identifications $(a, 0) \sim (a', 0)$ for all $a, a' \in A$, and $(a, 1) \sim i(a) = a$.

($\Longrightarrow$) Let $h : A \times I \to X$ be the null-homotopy of the inclusion such that $h(a, 0) = a_0$ and $h(a, 1) = a$ for some $a_0 \in A$. To construct the necessary retract, we can first define a map $\widetilde{r} : A \times I \coprod X \to X$ which is constant on equivalence classes, and thus descends to a retract $r : C_i \to X$. Let $\widetilde{r}$ be such that if $x \in X$, $\widetilde{r}(x) = x$. Otherwise, if $x = (a, t) \in A \times I, \widetilde{r}(x) = \widetilde{r}(a, t) = h(a, t)$. To verify that the descent map is well-defined, we check the two identifications. $\widetilde{r}(a, 0) = h(a, 0) = a_0 = h(a', 0) = \widetilde{r}(a', 0)$ and $\widetilde{r}(a, 1) = h(a, 1) = a = \widetilde{r}(a)$, so the descent map is well-defined. It is also a retract since $r(x) = x$ for all $x \in X$.

($\Longleftarrow$) Let $r : C_i \to X$ be a retract. This extends to a map $\widetilde{r} : A \times I \coprod X \to X$ which is constant on equivalence classes $(a, 1) \sim a$ and $(a, 0) = (a', 0)$ for all $a, a' \in A$. Hence, for all $a, a' \in A, \widetilde{r}(a, 0) = \widetilde{r}(a', 0)$, thus $\widetilde{r}(a, 0) = a_0$ for some $a_0 \in A$ for all $a \in A$. Additionally, $\widetilde{r}(a, 1) = \widetilde{r}(a)$, and since r is a retract, $\widetilde{r}(a) = a$. Recapitulating, for all $a \in A, \widetilde{r}(a, 0) = a_0$ and $\widetilde{r}(a, 1) = a$, so $\widetilde{r}|_{A \times I}$ is the desired null-homotopy of the inclusion $i : A \to X$.

Now, suppose that $i : A \to X$ is null-homotopic. Consider the long exact sequence of a pair

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots .$$

Since i is null-homotopic, $i_* = 0$ for all $H_n(A) \to H_n(X)$ for $n \geqslant 1$, and the identity for $n = 0$. By exactness, each j_* has trivial kernel, and thus is injective for $n \geqslant 1$. Switching to reduced homology thus gives short exact sequences

$$0 \to \widetilde{H}_n(X) \to H_n(X, A) \to \widetilde{H}_{n-1}(A) \to 0$$

for $n \geqslant 1$. By Hatcher, p. 125, $\widetilde{H}_n(C_i) \simeq H_n(X, A)$. By the previous argument, since the inclusion is null-homotopic, X must be a retract of C_i. By definition of retract, $r \circ i = \mathrm{id}_X$, so by functoriality $r_* \circ i_* = (\mathrm{id}_X)_*$, and thus the short exact sequence splits, yielding $H_n(X, A) \simeq \widetilde{H}_n(C_i) \simeq \widetilde{H}_n(X) \oplus \widetilde{H}_{n-1}(A)$ for each $n \geqslant 1$. ■

> **Exercise 94.** *Let X be path-connected, locally path-connected, and semi-locally simply connected. Let $G_1 \subset G_2$ be subgroups of $\pi_1(X, x_0)$. Let $p_i : X_{G_i} \to X$ be the covering map corresponding to G_i. There is a covering space map $f : X_{G_1} \to X_{G_2}$ such that $p_2 \circ f = p_1$.*

Proof. The setup of this problem corresponds to **Proposition 1.36** in Hatcher. In the construction, X_{G_1} and X_{G_2} are quotients of the universal cover, and hence are path-connected covering spaces. Indeed, the correspondence refers to isomorphism classes of *path-connected* covering spaces, **Theorem 1.38** in Hatcher. Additionally, X_{G_1} and X_{G_2} must be locally path-connected as well, since a covering map is a local homeomorphism and X is locally path-connected. Now, consider the solid diagram

$$\begin{tikzcd} & X_{G_2} \arrow[d, "p_2"] \\ X_{G_1} \arrow[r, "p_1"'] \arrow[ur, dashed, "f"] & X \end{tikzcd}.$$

Pick arbitrary basepoints $x_0' \in p_1^{-1}(x_0)$ and $x_0'' \in p_2^{-1}(x_0)$. Since p_2 is a covering map and X_{G_1} is path-connected and locally path-connected, and $(p_1)_*(\pi_1(X_{G_1}, x_0')) = G_1 \subset G_2 = (p_2)_*(\pi_1(X_{G_2}, x_0''))$ by assumption, so by the lifting criterion, **Proposition 1.33** in Hatcher, f exists making the diagram commute.

It remains to show that f is a covering map. By assumption, p_2 is a covering map, so let $\{U_x\}$ denote the cover of X by evenly covered neighborhoods. For each $x \in X_{G_2}$, let V_x denote the neighborhood of x that maps homeomorphically onto the evenly covered neighborhood U_x of $p_2(x)$. Since $p_2 \circ f$ is a covering map, $(p_2 \circ f)^{-1}(U_x)$ is a disjoint union of open sets, or sheets, in X_{G_1}. Note that, for any particular sheet V_x with $p_2(V_x) = U_x$, $(p_2 \circ f)^{-1}(p_2(V_x)) = f^{-1}(p_2^{-1}(p_2(V_x))) = f^{-1}(V_x)$ is thus a disjoint union of open sets. Finally, since $p_2 \circ f|_{f^{-1}(V_x)}$ is a homeomorphism onto U_x, and $p_2|_{V_x}$ is a homeomorphism onto U_x, $f|_{f^{-1}(V_x)}$ must be a homeomorphism onto V_x. Thus, f is a covering map, with $\{V_x\}$ providing the cover of X_{G_2} by evenly covered sets. ■

> **Exercise 95.** *Let X be the space obtained from the cube I^3 by identifying opposite sides via the map translating the face by a unit distance in the normal direction and twisting by one-half of a full rotation. Below is a presentation of the fundamental group and all homology groups of X.*

Proof. The cube I^3 is homemorphic to the disk D^3, so X is homeomorphic to D^3 with antipodal points on ∂D^3 identified, since the faces of the cube are rotated by π before identification. $\mathbb{R}\mathrm{P}^3$ can be formed by identifying antipodes on S^3. Furthermore, this identification can be restricted to identifying antipodes on the equator of S^3, which is homeomorphic to D^3. Hence, we have that X is homeomorphic to $\mathbb{R}\mathrm{P}^3$. First, $\pi_1(\mathbb{R}\mathrm{P}^3) \simeq \mathbb{Z}/2\mathbb{Z}$ since S^3 is both a universal cover and a double cover. By Hatcher, **Example 2.42**,

$$H_i(\mathbb{R}\mathrm{P}^3) \simeq \begin{cases} \mathbb{Z} & i = 0 \text{ or } i = n \text{ odd.} \\ \mathbb{Z}/2\mathbb{Z} & 0 < i < n, i \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

A presentation for $\mathbb{Z}$ is given by $\langle i \mid \rangle$. That is, a single generator with no relations. A presentation for $\mathbb{Z}/2\mathbb{Z}$ is given by $\mathbb{Z}/2\mathbb{Z} \simeq \langle i \mid i^2 = 0 \rangle$. ■

> **Exercise 96.** *Let $f : Y \to X$ be a fibration and let $\alpha : I \to X$ be a path from x to y. Apply the defining property of a fibration to the map $f^{-1}(x) \times \{0\} \to f^{-1}(x) \times I$ to show that α induces a map $\alpha_* : f^{-1}(x) \to f^{-1}(y)$. The homotopy class of α_* only depends on the homotopy class of α, and that in fact this construction yields a functor $\Gamma X \to$ **HTop**, where $\Gamma(X)$ is the fundamental groupoid of X. In particular, any two points in the same path-component of X have homotopy equivalent fibers.*

Proof. Construct from the path α a homotopy $a_t : f^{-1}(x) \to X$ given by $a_t(f^{-1}(x)) = \alpha(t)$. An initial lift $\widetilde{a}_0$ is given by the inclusion $f^{-1}(x) \hookrightarrow Y$. f is a fibration, so by the homotopy lifting property, there exists a lift $\widetilde{a}_t : f^{-1}(x) \times I \to Y$. This construction yields the usual fibration commutative diagram

$$\begin{tikzcd} f^{-1}(x) \arrow[r, hook, "\widetilde{a}_0"] \arrow[d, "f^{-1}(x)\times\{0\}"'] & Y \arrow[d, "f"] \\ f^{-1}(x) \times I \arrow[r, "a"'] \arrow[ur, "\widetilde{a}"'] & X \end{tikzcd}.$$

Note that $\widetilde{a}_1$ is a map $\alpha_* : f^{-1}(x) \to f^{-1}(y)$ by commutativity since $f(\widetilde{a}(f^{-1}(x), 1)) = a(f^{-1}(x), 1) = \alpha(1) = y$.

Suppose $\alpha \simeq \alpha'$ rel ∂I via $h(s,t) : I \times I \to X$. We aim to show $a_* \simeq a'_*$. h gives maps $\mathfrak{h}_{st} : f^{-1}(x) \to X$ given by $\mathfrak{h}_{st}(f^{-1}(x)) = h(s,t)$. Let $\widetilde{h}_{0,t} = a_*$ and $\widetilde{h}_{1,t} = a'_*$. Let $\widetilde{h}_{s,0}$ be the inclusion $f^{-1}(x) \hookrightarrow Y$. These maps define an initial lift $\widetilde{h}_{\partial I} : f^{-1}(x) \times I \times \partial I \to Y$.

By Hatcher, p.405, f satisfies the homotopy lifting property for any pair $(Z \times I, Z \times \partial I)$. Thus there exists $\widetilde{h}$ such that

$$\begin{tikzcd} f^{-1}(x) \times I \times \partial I \arrow[r, "\widetilde{h}_{\partial I}"] \arrow[d, hook] & Y \arrow[d, "f"] \\ f^{-1}(x) \times I \times I \arrow[r, "\mathfrak{h}"'] \arrow[ur, "\widetilde{h}"'] & X \end{tikzcd}$$

commutes. $\widetilde{h}_{s,1}$ then gives a homotopy a_* to a'_* by commutativity. Hence, the homotopy class of each a_* is independent of the choice of lift. It then follows that this construction is a well-defined map $F : \Gamma X \to$ **HTop**. Even better, F is actually a functor, as it preserves identity morphisms and composition of morphisms. It preserves the identity since a constant path α_x at x gives a homotopy a_* to a_*, which is the identity homotopy. It also preserves composition since for the composition of two paths $a * a'$, the lift $(a * a')_*$ is homotopy equivalent to $a_* a'_*$. To see this, if $\widetilde{a}_t$ is a lift where $\widetilde{a}_1 = a_*$ and $\widetilde{a}'_t$ is defined similarly, then let $\widetilde{\mathfrak{a}}_t$ be defined as $\widetilde{a}_{2t}$ for $t \in [0, \frac{1}{2}]$ and $\widetilde{a}'_{2t-1} \circ a_*$ for $t \in [\frac{1}{2}, 1]$. This gives a lift $\widetilde{\mathfrak{a}}_t$ where $\widetilde{\mathfrak{a}}_1 = (a * a')_*$.

Finally, if $\alpha : I \to X$ is a path from x to y, then $F(\alpha)$ is a homotopy $f^{-1}(x)$ to $f^{-1}(y)$ with inverse $F(\alpha^{-1})$, since $\mathrm{id} = F(\mathrm{id}) = F(\alpha^{-1} * \alpha) = F(\alpha^{-1}) \circ F(\alpha)$ by functoriality. Thus a path x to y gives a homotopy equivalence $f^{-1}(x)$ to $f^{-1}(y)$. ∎

> **Exercise 97.** *Let X be a space. $H_n(X;\mathbb{Z}) = 0$ for all $n > 0$ if, and only if, $H_n(X;\mathbb{Q}) = 0$ and $H_n(X;\mathbb{Z}/p\mathbb{Z}) = 0$ for all prime numbers p and all $n > 0$.*

Proof. By the universal coefficient theorem, the sequence

$$0 \to H_i(X;\mathbb{Z}) \otimes A \to H_i(X;A) \to \mathrm{Tor}(H_{i-1}(X;\mathbb{Z}), A) \to 0$$

is exact for any abelian group A.

($\implies$) If $H_n(X;\mathbb{Z}) = 0$ for all $n > 0$, the sequence above becomes

$$0 \to H_n(X;A) \to \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A) \to 0$$

for all $n > 0$, since $0 \otimes A = 0$ as any $0 \otimes a$ is the zero element by Hatcher, p. 215. Thus, by exactness, $H_n(X;A) \simeq \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A)$. For $n > 1, \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A) = 0$ since $H_{n-1}(X;\mathbb{Z}) = 0$ and 0 is torsion-free. Additionally, for $n = 1$, $H_0(X;\mathbb{Z})$ counts the path-components of X, so it is isomorphic to $\mathbb{Z}^n$ for some n, which is also torsion-free. Hence, $\mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A) = 0$ for all $n > 0$ by **Proposition 3A.5** in Hatcher and thus $H_n(X;A) = 0$ for $n > 0$ for both $A = \mathbb{Q}$ and $A = \mathbb{Z}/p\mathbb{Z}$ for all prime p.

($\impliedby$) If $H_n(X,\mathbb{Q}) = 0$ for all $n > 0$, setting $A = \mathbb{Q}$, the sequence above yields that

$$0 \to H_n(X;\mathbb{Z}) \otimes \mathbb{Q} \to 0 \to \mathrm{Tor}(H_{i-1}(X;\mathbb{Z}), \mathbb{Q}) \to 0$$

is exact, so by exactness, $H_n(X;\mathbb{Z}) \otimes \mathbb{Q} = 0$. We claim $H_n(X;\mathbb{Z}) \simeq H_n(X;\mathbb{Z}) \otimes \mathbb{Q}$, which then finishes the proof as $H_n(X;\mathbb{Z}) \otimes \mathbb{Q} = 0$.

$H_n(X;\mathbb{Z}/p\mathbb{Z}) = 0$ for all $n > 0$ and all primes p if and only if $H_n(X;\mathbb{Z})$ is a vector space over $\mathbb{Q}$ for all $n > 0$. Also, by Hatcher, p. 215, $H_n(X) \otimes \mathbb{Q} = H_n(X) \otimes_{\mathbb{Q}} \mathbb{Q}$. This reduces what is left to showing that for X a vector space over $\mathbb{Q}$, $X \otimes_{\mathbb{Q}} \mathbb{Q} \simeq X$. But it is a general algebraic fact that if M is an R-module, then $M \otimes_R R \simeq M$. ∎

> **Exercise 98.** *If X and Y are pointed spaces and $n \geqslant 2$,*
> $$\pi_n(X \vee Y) \simeq \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

Proof. Consider the long exact sequence of relative homotopy groups

$$\ldots \to \pi_{n+1}(X \times Y, X \vee Y) \xrightarrow{\partial} \pi_n(X \vee Y) \xrightarrow{i_*} \pi_n(X \times Y) \to \ldots$$

where i_* is the induced map of the inclusion $i : X \vee Y \to X \times Y$. We claim that there is a right splitting map j_* such that $i_* \circ j_* = \mathrm{id}_*$. This claim gives that $\pi_n(X \vee Y) \simeq \pi_n(X \times Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$, from which the proposition follows by **Proposition 4.2** in Hatcher, which gives $\pi_n(X \times Y) \simeq \pi_n(X) \times \pi_n(Y)$.

Now to address the claim. Consider any element $[\gamma] \in \pi_n(X \times Y)$, represented by $\gamma : S^n \to X \times Y$. Composing with the natural projections p_x and p_y gives $p_x \circ \gamma : S^n \to X$ and $p_y \circ \gamma : S^n \to Y$, which when composed with the inclusion give $i \circ p_x \circ \gamma : S^n \to X \hookrightarrow X \vee Y$ and $i \circ p_y \circ \gamma : S^n \to Y \hookrightarrow X \vee Y$. Re-label these γ^x and γ^y respectively. Now consider the equivalence class $[\gamma^x * \gamma^y] \in \pi_n(X \vee Y)$. Define a map $j_*([\gamma]) = [\gamma^x * \gamma^y]$. It remains to show that j_* is a homomorphism and that $i_* \circ j_* = \mathrm{id}_*$.

The claim that j_* is a homomorphism will require that π_n is abelian, so consider $n \geqslant 2$. Note that the projection and inclusions induce homomorphisms, so $(\gamma_1 * \gamma_2)^x = \gamma_1^x * \gamma_2^x$, and similarly for y. Explicitly computing,

$$\begin{aligned} j_*([\gamma_1][\gamma_2]) &= j_*([\gamma_1 * \gamma_2]) \\ &= [(\gamma_1 * \gamma_2)^x * (\gamma_1 * \gamma_2)^y] \\ &= [\gamma_1^x * \gamma_2^x * \gamma_1^y * \gamma_2^y] \\ &= [\gamma_1^x * \gamma_1^y * \gamma_2^x * \gamma_2^y] \\ &= [\gamma_1^x * \gamma_1^y][\gamma_2^x * \gamma_2^y] \\ &= j_*([\gamma_1])j_*([\gamma_2]) \end{aligned}$$

Now, to prove $i_* \circ j_* = \mathrm{id}_*$, we show a homotopy $i(\gamma_1^x * \gamma_1^y) \simeq \gamma$, which gives that $i_*(j_*([\gamma]) = i_*([\gamma_1^x * \gamma_1^y]) = [\gamma]$. Rewriting, $\gamma = (p_x(\gamma_1^x), p_y(\gamma_1^y))$, and $i(\gamma_1^x * \gamma_1^y) = (p_x(\gamma_1^x), y_0) * (x_0, p_y(\gamma_1^y))$, so the explicit homotopy $\gamma \mapsto \gamma_1^x * \gamma_1^y$ is given by the usual reparametrizing homotopy used to show that π_n is a group. ■