TOPOLOGY NOTES

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Preamble. The following is a collection of exercises relating to point-set topology and preliminary algebraic topology, together with my proofs of those exercises. Use at your own risk.

Proposition 1. *$S = (-\infty, a) \cup (b, \infty)$ for fixed $a < b \in \mathbb{R}$ is open. $\mathbb{R} \backslash S$ is not open.*

Proof. For any point $s \in S$, $B_\delta(s) \subset S$ for $\delta = \min(|s-a|, |s-b|)$. Its complement is not open since any $B_\delta(a)$ must contain points in S for all $\delta > 0$. ∎

Proposition 2. *$\mathbb{Z}$ is not open. $\mathbb{R}\backslash\mathbb{Z}$ is open.*

Proof. $\mathbb{Z}\backslash\mathbb{R} = \bigcup_{i \in \mathbb{Z}} (i, i+1)$ and hence is open since arbitrary unions of open sets are open. $\mathbb{Z}$ itself is not open since any nonempty, open subset of $\mathbb{R}$ contains rational numbers. ∎

Proposition 3. *$\mathbb{Q}$ is not open nor is $\mathbb{R}\backslash\mathbb{Q}$.*

Proof. The rationals and irrationals are dense in $\mathbb{R}$. Hence, for any $x \in \mathbb{Q}$, $B_\delta(x)$ must contain an element in $\mathbb{R}\backslash\mathbb{Q}$ for all $\delta > 0$, and therefore $\mathbb{Q}$ is not open. Similarly, its complement also cannot be open. ∎

Proposition 4. *$S = \{1/n \mid n \in \mathbb{Z}^+\}$ is not open nor is $\mathbb{R}\backslash S$.*

Proof. For all $s \in S$, $B_\delta(s)$ contains irrational numbers for all $\delta > 0$, and hence S cannot be open. The complement of S is also not open, since any $B_\delta(0)$ must contain some $1/n$ for any $\delta > 0$ (consider $n > 1/\delta$) . ∎

Proposition 5. *$f(x) = |x|$ is continuous on $\mathbb{R}$.*

Proof. For all $\epsilon > 0$, take $\delta = \epsilon$. We have that for all $x_0 \in \mathbb{R}$, $|x - x_0| < \epsilon$ implies $|f(x) - f(x_0)| = ||x| - |x_0|| \leqslant |x - x_0| < \epsilon$ and hence f is continuous. ∎

Proposition 6. $g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$ *is not continuous on $\mathbb{R}$.*

Proof. The preimage $g^{-1}(B_{1/2}(0)) = \mathbb{Q}$ is not open by **Proposition 3**. ∎

Proposition 7. *$f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(V)$ is closed for any closed $V \subset \mathbb{R}$.*

Proof. If f is continuous, then $f^{-1}(\mathbb{R}\backslash V) = \mathbb{R}\backslash f^{-1}(V)$ is open and hence $f^{-1}(V)$ is closed. The other direction follows similarly. ∎

Proposition 8. *The image of any open set is not necessarily open for a continuous function.*

Proof. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$. The image of any open set is $\{0\}$. ∎

Proposition 9. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open, then so is $U \times V \subset \mathbb{R}^{m+n}$.*

Proof. Given $(u, v) \in U \times V$, we must have $B_{\delta_1}(u) \subset U$ and $B_{\delta_2}(v) \subset V$, hence $B_{\delta_1}(u) \times B_{\delta_2}(v) \subset U \times V$. Taking $\delta = \min(\delta_1, \delta_2)$ gives $B_\delta((u, v)) \subset B_{\delta_1}(u) \times B_{\delta_2}(v) \subset U \times V$. ∎

Proposition 10. *The open disk $D_1 = \{(x, y) \mid x^2 + y^2 < 1\}$ cannot be written as the Cartesian product of two open sets $U, V \subset \mathbb{R}$.*

Proof. Suppose that $D_1 = U \times V$ for some open $U, V \subset \mathbb{R}$. Since $(0, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, 0) \in U \times V$, we must have $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \in U \times V$, but this point is not in D_1. ∎

Proposition 11. *Let $S = \bigcup_{i=1}^{n} L_i \subset \mathbb{R}^2$ for $n \in \mathbb{N}$ be the union of a finite number of lines L_i. $\mathbb{R} \backslash S$ is open.*

Proof. We proceed by induction on the number of lines. If $S = L$ for some line L, for all points $x \in \mathbb{R}\backslash S$, take δ to be the perpendicular distance from x to L. Otherwise, suppose $\mathbb{R}\backslash S$ is open for $S = \bigcup_{i=1}^{n} L_i$. For all $x \in \mathbb{R}\backslash S$, there exists $\delta_s > 0$ such that $B_{\delta_s}(x) \subset \mathbb{R}\backslash S$. For the set $(\mathbb{R}\backslash S)\backslash L_{i+1}$, take δ to be the minimum of δ_s and the perpendicular distance to L_{i+1}. We must have that $B_\delta(x) \subset (\mathbb{R}\backslash S)\backslash L_{i+1}$ and hence $R\backslash S$ is open for $S = \bigcup_{i=1}^{n+1} L_i$. ∎

Proposition 12. *Let X and Y be sets. A function $f : X \to Y$ is continuous for every topology $\mathcal{T}$ on X and every topology $\mathcal{S}$ on Y if and only if f is constant.*

Proof. If $X = \varnothing$, then f is vacuously constant. Otherwise, there exists $x \in X$. Consider $\mathcal{T} = \{\varnothing, X\}$ and $\mathcal{S} = \mathcal{P}(Y)$. If f is continuous, $f^{-1}(\{f(x)\}) \in \mathcal{T}$ since $\{f(x)\} \in \mathcal{S}$, and hence $f^{-1}(\{f(x)\}) = X$, thus f is constant .

If f is constant, the preimage of any open set is $\varnothing$ or X, and thus open, hence f is continuous. ∎

Proposition 13. *Let X be a set. $\mathcal{T} = \{U \subset X \mid U = \varnothing \vee |X\backslash U| \in \mathbb{N}\}$ is a topology on X.*

Proof. Note $\varnothing, X \in \mathcal{T}$. For Λ any index set, $U_\lambda \in \mathcal{T}$, we have $X\backslash \bigcup_{\lambda\in\Lambda} U_\lambda = \bigcap_{\lambda\in\Lambda} X\backslash U_\lambda$, an intersection of finite sets, which must be finite. For finite Λ, $X\backslash \bigcap_{\lambda\in\Lambda} U_\lambda = \bigcup_{\lambda\in\Lambda} X\backslash U_\lambda$, a finite union of finite sets, which must be finite. ∎

Proposition 14. *For X equipped with cofinite topology, $f : X \to X$ is continuous if and only if $f^{-1}(\{x\})$ is finite for all $x \in X$ or f is constant.*

Proof. If $X = \varnothing$, the proposition follows vacuously. Otherwise, there exists $x \in X$. If f is continuous, since $\{x\}$ is closed, $f^{-1}(\{x\})$ is closed, hence either X or finite.

If f is constant, then it is continuous. Otherwise, consider any closed V. It must be $V = \bigcup_{\lambda\in\Lambda}\{x_\lambda\}$ for Λ a finite index set. We have $f^{-1}(V) = f^{-1}(\bigcup_{\lambda\in\Lambda}\{x_\lambda\}) = \bigcup_{\lambda\in\Lambda} f^{-1}(\{x_\lambda\})$, a finite union of finite sets, hence finite and closed. ∎

Proposition 15. 1 *in the following table indicates when the identity map is continuous for various topologies on* $\mathbb{R}$, 0 *indicates otherwise. Map is from row label to column label.*

$i : \mathbb{R} \to \mathbb{R}$	Discrete	Standard	Cofinite	Indiscrete
Discrete	1	1	1	1
Standard	0	1	1	1
Cofinite	0	0	1	1
Indiscrete	0	0	0	1

Proof. For any open set U, $i^{-1}(U) = U$, hence the table follows since Indiscrete $\subset$ Cofinite $\subset$ Standard $\subset$ Discrete. ∎

Proposition 16. *Let $\mathcal{T}$ be subsets $S \subset \mathbb{R}$ such that, for all $x \in S$, there exists $a, b \in \mathbb{R}$ such that $x \in [a, b) \subset S$. $\mathcal{T}$ is a topology.*

Proof. Note $\{\varnothing, \mathbb{R}\} \subset \mathcal{T}$. For Λ any index set, $S_\lambda \in \mathcal{T}$, $x \in \bigcup_{\lambda \in \Lambda} S_\lambda$ implies $x \in S_\lambda$ for some $\lambda \in \Lambda$. Since S_λ is open, there exist a, b such that $x \in [a, b) \subset S_\lambda \subset \bigcup_{\lambda \in \Lambda} S_\lambda$, hence the union is open. Finally, it suffices to show the intersection of two open sets is open, since the general case follows by induction. For open S_1, S_2, $x \in S_1 \cap S_2$ implies $x \in [a_1, b_1) \subset S_1$ and $x \in [a_2, b_2) \subset S_2$. Thus $x \in [\max(a_1, a_2), \min(b_1, b_2)) \subset S_1 \cap S_2$, hence the intersection is open. ∎

Proposition 17. *The standard topology on $\mathbb{R}$ induces the discrete topology on $\mathbb{Z}$.*

Proof. It suffices to show for all $U \in \mathcal{P}(\mathbb{Z})$, there exists an open set $V \subset \mathbb{R}$ such that $U = \mathbb{Z} \cap V$, since the induced topology must be coarser than $\mathcal{P}(\mathbb{Z})$. Let $V = \bigcup_{z \in U} B_{\frac{1}{7}}(z)$. V is open since it is the union of open balls, and $\mathbb{Z} \cap V = U$. ∎

Proposition 18. *Identify $\mathbb{R}$ with $\{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. The standard topology on $\mathbb{R}^2$ induces the standard topology on $\mathbb{R}$.*

Proof. For all $U \subset \mathbb{R}$ open, there exists $V \subset \mathbb{R}^2$ open such that $U = V \cap \mathbb{R}$, namely $V = U \times \mathbb{R}$, hence the induced topology is finer than the standard topology. Any open set in the standard topology on $\mathbb{R}^2$ can be written as $U = \bigcup_{\lambda \in \Lambda} B_{\delta_\lambda}(x_\lambda)$ for Λ some index set. We have $\mathbb{R} \cap U = \mathbb{R} \cap \bigcup_{\lambda \in \Lambda} B_{\delta_\lambda}(x_\lambda) = \bigcup_{\lambda \in \Lambda} \mathbb{R} \cap B_{\delta_\lambda}(x_\lambda)$. Since each $\mathbb{R} \cap B_{\delta_\lambda}(x_\lambda)$ is either empty or an open interval, and the union of open intervals is open in the standard topology on $\mathbb{R}$, the standard topology must be finer than the induced topology. ∎

Proposition 19. *For $(X, \mathcal{T})$ a topological space, Y a set, $f : X \to Y$ any function, $\mathcal{S} = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}\}$ is a topology on Y.*

Proof. Note $\varnothing, Y \in \mathcal{S}$. For Λ any index set, $U_\lambda \in \mathcal{S}$, we have $f^{-1}(\bigcup_{\lambda \in \Lambda} U_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) \in \mathcal{T}$ since each $f^{-1}(U_\lambda) \in \mathcal{T}$ and $\mathcal{T}$ is closed under union. Closure under intersection follows similarly with Λ finite. ∎

Proposition 20. *In the product of two topological spaces $Y \times Z$, a subset U is open if and only if it can be expressed as the union $\bigcup_{\lambda \in \Lambda} V_\lambda \times W_\lambda$ for some open $V_\lambda \subset Y$ and $W_\lambda \subset Z$.*

Proof. The proposition follows since products of open sets form a basis for the product topology, and any open set in the generated topology of a basis can be expressed as a union of basis elements. ∎

Proposition 21. *Let X, Y, Z be topological spaces and let $Y \times Z$ have the product topology. $F = (f_1, f_2) : X \to Y \times Z$ is continuous if and only if f_1 and f_2 are continuous.*

Proof. If F is continuous, for all open $V \subset Y, F^{-1}(V \times Z) = f_1^{-1}(V) \cap f_2^{-1}(Z) = f_1^{-1}(V) \cap X = f^{-1}(V)$ is open, hence f_1 is continuous. Similarly, f_2 must also be continuous.

If f_1 and f_2 are continuous, then for all open $U \subset Y \times Z$, we have $F^{-1}(U) = F^{-1}(\bigcup_{\lambda \in \Lambda} V_\lambda \times W_\lambda) = \bigcup_{\lambda \in \Lambda} F^{-1}(V_\lambda \times W_\lambda) = \bigcup_{\lambda \in \Lambda} f_1^{-1}(V_\lambda) \cap f_2^{-1}(W_\lambda)$ is open since the intersection of any two open sets is open, and an arbitrary union of open sets is open. ∎

Proposition 22. $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ *with the subspace topology from* $\mathbb{R}$ *is homeomorphic to* $\mathbb{R}$.

Proof. We use $f : \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = e^x$ with continuous inverse $f^{-1} = \ln(x)$ as our homeomorphism. ∎

Proposition 23. *Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ equipped with the subspace topology. $\mathbb{R}^2 \backslash \{0\}$ with subspace topology is homeomorphic to $\mathbb{R} \times S$ with the product topology.*

Proof. Let $f_1 : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ be given by $f_1(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$. Let $f_2 : \mathbb{R}^2 \backslash \{0\} \to S$ be given by $f_2(x, y) = \left(x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2}\right)$. Let $g = (f_1, f_2)$. We know from analysis f_1 and f_2 are continuous, hence by **Proposition 3**, g is also continuous. $g^{-1}(w, u, v) = (e^w u, e^w v)$ is also continuous, hence we have a homeomorphism. ∎

Proposition 24. *For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, define*

$$d(\mathbf{u}, \mathbf{v}) = \begin{cases} \|\mathbf{u} - \mathbf{v}\| & \text{if } \mathbf{u} = t\mathbf{v} \text{ for some } t \in \mathbb{R} \\ \|\mathbf{u}\| + \|\mathbf{v}\| & \text{otherwise} \end{cases}$$

d is a metric, but does not induce the standard topology.

Proof. Note $d(\mathbf{u}, \mathbf{v}) \geqslant 0$ and $d(\mathbf{u}, \mathbf{v}) = 0$ implies $\mathbf{u} = \mathbf{v}$ and $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v}$. If $\mathbf{x} = t\mathbf{y} = r\mathbf{z}$ for some $t, r \in \mathbb{R}$, then $d(\mathbf{x}, \mathbf{z}) \leqslant d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ since $\|\mathbf{x} - \mathbf{z}\| \leqslant \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|$. If t exists but r does not, then still $d(\mathbf{x}, \mathbf{z}) \leqslant d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ since $\|\mathbf{x}\| + \|\mathbf{z}\| \leqslant \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| + \|\mathbf{z}\|$. The other case where r exists but t does not is similar. If all points lie on different railway tracks, then $d(\mathbf{x}, \mathbf{z}) \leqslant d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ since $\|\mathbf{x}\| + \|\mathbf{z}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{y}\| + \|\mathbf{z}\|$.

The metric topology on $\mathbb{R}^2$ induced by d is not the standard topology, however, since $B_1(0, 1)$ is a line segment in the metric topology, which is not open in the standard topology. ∎

Proposition 25. *The topology $\{\{a,b\},\{a\},\emptyset\}$ on $\{a,b\}$ cannot come from any metric.*

Proof. We note this space is not Hausdorff since the points a and b do not have disjoint neighborhoods. ■

Proposition 26. *The metric topology on any finite set is the discrete topology.*

Proof. Each singleton set must be open for each point to have a disjoint neighborhood with all other points. Since the singletons generate the discrete topology, we are done. ■

Proposition 27. *For $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, define*

$$d'(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{n} |u_i - v_i|$$

d' is a metric inducing the standard topology.

Proof. Note $d'(\mathbf{u},\mathbf{v}) \geqslant 0$ and $d'(\mathbf{u},\mathbf{v}) = 0$ implies $\mathbf{u} = \mathbf{v}$ and $d'(\mathbf{u},\mathbf{v}) = d'(\mathbf{v},\mathbf{u})$ for all $\mathbf{u},\mathbf{v}$. Additionally, $d'(\mathbf{x},\mathbf{z}) = \sum_{i=1}^n |x_i - z_i| = \sum_{i=1}^n |x_i - y_i + y_i - z_i| \leqslant \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i| = d'(\mathbf{x},\mathbf{y}) + d'(\mathbf{y},\mathbf{z})$.

Let $\mathcal{S}$ be the set of basis elements for the standard topology and let $\mathcal{T}$ be the set of basis elements for the topology induced by d'. For $B_{\delta,d'}(x) \in \mathcal{T}$, note $\mathcal{S} \ni B_{\delta/\sqrt{n}}(x) \subset B_{\delta,d'}(x)$ by Cauchy-Schwartz. Similarly, for $B_\delta(x)$, we have $B_{\delta,d'}(x) \subset B_\delta(x)$. Hence, each topology is finer than the other, and thus they must be equal. ■

Proposition 28. *Let X, Y be topological spaces, $A \subset X$, $B \subset Y$. Then for $X \times Y$ with the product topology, $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Proof. ($\supset$) Let $(a,b) \in \overline{A} \times \overline{B}$, $W \subset X \times Y$ open, $(a,b) \in W$. Consider a basis element $U \times V \subset W$ with $a \in U$, $b \in V$. Since $a \in \overline{A}$ and $b \in \overline{B}$, $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. Hence $U \times V \cap A \times B \neq \emptyset$, so $(a,b) \in \overline{A \times B}$. ($\subset$) $A \times B \subset \overline{A} \times \overline{B}$ implies $\overline{A \times B} \subset \overline{\overline{A} \times \overline{B}} = \overline{A} \times \overline{B}$. ■

Proposition 29. *Let $a < b < c < d < e$, all in $\mathbb{R}$. Then no two of $A \cap B, \overline{A} \cap B, A \cap \overline{B}, \overline{A \cap B}, \overline{A} \cap \overline{B}$ are equal for $A = (a,c) \cup (d,e)$ and $B = [a,b) \cup \{d\}$.*

Proof. Explicitly computing gives

$$\begin{aligned} A \cap B &= (a,b) \\ \overline{A} \cap B &= [a,b) \cup \{d\} \\ A \cap \overline{B} &= (a,b] \\ \overline{A \cap B} &= [a,b] \\ \overline{A} \cap \overline{B} &= [a,b] \cup \{d\} \end{aligned}$$

■

Proposition 30. *If X is Hausdorff, then $\{x\}$ is closed for all $x \in X$.*

Proof. For all $y \in X\backslash\{x\}$, theres exists an open neighborhood $B(y)$ such that $y \in B(y)$ but $x \notin B(y)$, so $X\backslash\{x\} = \bigcup_{y \in X\backslash\{x\}} B(y)$ is open, hence $\{x\}$ is closed. ■

Proposition 31. *X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\} \subset X \times X$ is closed.*

Proof. ($\Longrightarrow$) If distinct points x, y have disjoint neighborhoods U, V, then $X \times X\backslash\Delta$ is open since $(x,y) \in U \times V$ and $U \times V \cap \Delta = \varnothing$ since U, V are disjoint. ($\Longleftarrow$) $X \times X\backslash\Delta$ open implies for all (x,y), there exists open W s.t. $(x,y) \in W \subset X \times X\backslash\Delta$. Since W is open in the product topology, there exist U, V open s.t. $(x,y) \in U \times V \subset W$. $W \cap \Delta = \varnothing$ implies U and V are disjoint, hence X is Hausdorff. ■

Proposition 32. *Let $f : X \to Y$ be continuous, $C \subset Y$ closed, and $D \subset X$ dense. Then $f(D) \subset C$ implies $f(X) \subset C$.*

Proof. f continuous implies $f^{-1}(Y\backslash C)$ is open. If $f^{-1}(Y\backslash C)$ is nonempty, then there exists $d \in f^{-1}(Y\backslash C) \cap D$ since D is dense, and $f(d) \notin C$, a contradiction to $f(D) \subset C$. Hence $f^{-1}(Y\backslash C)$ is empty, so $f(X) \subset C$. ■

Proposition 33. *Let X, Y be topological spaces with Y Hausdorff. Let $f, g : X \to Y$ be continuous functions. If $D \subset X$ is dense and $f|_D = g|_D$, then $f = g$.*

Proof. Consider $h : X \to Y \times Y$ given by $h(x) = (f(x), g(x))$. Since f, g are continuous, h is continuous. By **Proposition 4**, Y is Hausdorff implies Δ is closed. $f|_D = g|_D$ gives $h(D) \subset \Delta$, so $h(X) \subset \Delta$ by **Proposition 32**. Hence $f = g$. ■

Proposition 34. *For X a topological space, $A \subset X$ a subset, A has no limit points in itself if and only if the subspace topology on A is discrete.*

Proof. ($\Longrightarrow$) For all $a \in A$, a not a limit point of A implies there is an open neighborhood $U \ni a$ such that $U \cap A = \{a\}$, hence $\{a\}$ is open in the subspace topology. Since singletons generate the discrete topology, we're done. ($\Longleftarrow$) Subspace topology being discrete implies that for all $a \in A$, there exists U open in X such that $U \cap A = \{a\}$, hence a is not a limit point of A. ■

Proposition 35. *Let $S = \{1/n \mid n \in \mathbb{Z}^+\} \subset \mathbb{R}$ with the standard topology. S has discrete subspace topology and a limit point outside itself.*

Proof. Given $n \in \mathbb{Z}^+$, let $\delta = \frac{1}{n(n+1)}$. Then $B_\delta(1/n) \cap S = \{1/n\}$, so the subspace topology on S is discrete. But any $B_\delta(0)$ contains some $1/n$ for $n > 1/\delta$, so 0 is a limit point of S, yet $0 \notin S$. ■

Proposition 36. *Let X, Y be topological spaces. Let V_1 and V_2 be open subsets of X s.t. $V_1 \cup V_2 = X$. Let $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ be functions s.t. $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$. Then $f : X \to Y$ given by*

$$f(x) = \begin{cases} f_1(x) & x \in V_1 \\ f_2(x) & x \in V_2 \end{cases}$$

is continuous if and only if both f_1 and f_2 are continuous.

Proof. Since $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$, f is well defined. If f is continuous, then for any open $U \subset Y$, we have $f_1^{-1}(U) = (f|_{V_1})^{-1}(U) = V_1 \cap f^{-1}(U)$, an intersection of open sets, and thus open. Hence f_1 must be continuous. Continuity of f_2 follows similarly.

If f_1 and f_2 are continuous, then for any open $U \subset Y$, we have $f^{-1}(U) = (f|_X)^{-1} = (f|_{V_1 \cup V_2})^{-1} = (f|_{V_1})^{-1}(U) \cup (f|_{V_2})^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$, a union of two open sets, and hence open. Thus f is continuous. ■

Proposition 37. *Let $\{A_n | n \in \mathbb{N}\}$ be a sequence of connected subsets of X such that for each n, $A_n \cap A_{n+1} \neq \varnothing$. Then $\bigcup_{n=0}^{\infty} A_n = S$ is connected.*

Proof. Let $U \subset S$ be nonempty, clopen. Then there exists $i \in \mathbb{N}$ s.t. $A_i \cap U \neq \varnothing$. Since A_i is connected and $A_i \cap U$ is a nonempty clopen subset of A_i, we have $A_i = A_i \cap U$ and hence $A_i \subset U$. Then $A_{i-1} \cap U \neq \varnothing$ and $A_{i+1} \cap U \neq \varnothing$. By the same reasoning, $A_{i-1} \subset U$ and $A_{i+1} \subset U$. By induction both ways, it follows that for all $i \in \mathbb{N}, A_i \subset U$. Hence $S \subset U$, and thus $S = U$. ■

Proposition 38. *If $A \subset X$, let the boundary of A be $BdA = \overline{A} \backslash A^\circ$, the closure minus the interior. X is connected if and only if every proper nonempty subset has nonempty boundary.*

Proof. If $\mathrm{Bd}A = \overline{A} \backslash A^\circ = \varnothing$, then $\overline{A} \subset A^\circ$, giving $A = \overline{A} = A^\circ$ since $A^\circ \subset A \subset \overline{A}$, implying A is clopen. If X is connected, then any proper, nonempty subset is not clopen, hence its boundary is nonempty.

If every proper, nonempty subset A has nonempty boundary, then $\overline{A} \neq A^\circ$, and hence A is not clopen, so the only clopen subsets are X and $\varnothing$, hence X is connected. ■

Proposition 39. *No two of* $(0, 1), (0, 1], [0, 1]$ *are homeomorphic.*

Proof. By Heine-Borel, $[0, 1]$ is compact and $(0, 1), (0, 1]$ are not compact. Hence $[0, 1]$ cannot be homeomorphic to either of the other two sets since the image of a compact set under a continuous function is compact.

Suppose $f : (0, 1] \to (0, 1)$ is a homeomorphism. Since f is injective, $f((0, 1]\backslash\{1\}) = f((0, 1])\backslash\{f(1)\}$. Since f is surjective, $f((0, 1])\backslash\{f(1)\} = (0, 1)\backslash\{f(1)\}$. f continuous implies $f((0, 1)) = (0, 1)\backslash\{f(1)\}$ is connected. But $(0, f(1)) \cup (f(1), 1) = (0, 1)\backslash\{f(1)\}$ and $(0, f(1)) \cap (f(1), 1) = \varnothing$, so $f((0, 1))$ is not connected, a contradiction. ■

Proposition 40. $\mathbb{R}^n\backslash\{0\}$ *is connected.*

Proof. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let $\pi_1(x) = x_1$. Let $U = \{x \in \mathbb{R}^n\backslash\{0\} \mid \pi_1(x) < 0\}$ and $V = \{x \in \mathbb{R}^n\backslash\{0\} \mid \pi_1(x) > 0\}$. U and V are convex, and thus connected. This implies their closures are connected. Since $\overline{U}$ and $\overline{V}$ are not disjoint, the union $\overline{U} \cup \overline{V} = \mathbb{R}^n\backslash\{0\}$ is connected. ■

Proposition 41. $\mathbb{R}$ *is not homeomorphic to* $\mathbb{R}^n$ *for* $n > 0$*.*

Proof. We proceed as in **Proposition 39**. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism. Then $f(\mathbb{R}^n\backslash\{0\}) = f(\mathbb{R}^n)\backslash\{f(0)\} = \mathbb{R}\backslash\{f(0)\}$ is connected by **Proposition**

40. But $(-\infty, f(0)) \cup (f(0), \infty) = \mathbb{R}\backslash\{f(0)\}$ and $(-\infty, f(0)) \cap (f(0), \infty) = \varnothing$, a contradiction. ■

Proposition 42. *The cardinality of the set of lines through any point in $\mathbb{R}^2$ is at least uncountable.*

Proof. For any point $x = (x_0, y_0) \in \mathbb{R}^2$, let $L = \{m = \frac{y-y_0}{x-x_0} \mid m \in \mathbb{R}\}$. Since $\mathbb{R}$ is uncountable, L is uncountable. But every line in L passes through x, so the number of lines passing through x must be at least as large in cardinality as L. ■

Proposition 43. *The complement of any countable set in $\mathbb{R}^2$ is path-connected, and hence connected.*

Proof. Consider a countable set $S \subset \mathbb{R}^2$ and any two points x, y in $\mathbb{R}^2\backslash S$. Let L_x and L_y denote the set of all lines passing through x and y respectively. By **Proposition 42**, these sets are at least uncountable.

Let $L_x^c = \{m = \frac{y_1-y_0}{x_1-x_0} \mid (x_0, y_0) = x, (x_1, y_1) \in S\}$ be the set of lines in L_x passing through a point in S. Define L_y^c similarly. Since S is countable, L_x^c and L_y^c are countable. Hence $L_x\backslash L_x^c$ and $L_y\backslash L_y^c$ are at least uncountable. Let L_1 be a line in $L_x\backslash L_x^c$ with slope m_1. Since $L_y\backslash L_y^c$ is uncountable, we can find L_2 in $L_y\backslash L_y^c$ with slope $m_2 \neq m_1$. Therefore L_1 and L_2 intersect by the parallel postulate. Let x_c denote the point of intersection. Then $\gamma : [0, 1] \to \mathbb{R}^2$ given by

$$\gamma(t) = \begin{cases} (1-2t)x + 2tx_c & t \in [0, \frac{1}{2}] \\ 2(1-t)x_c + (2t-1)y & t \in [\frac{1}{2}, 1] \end{cases}$$

is a continuous function by the gluing lemma, with $\gamma(0) = x$ and $\gamma(1) = y$, hence a path from x to y. Additionally, $\gamma([0, 1]) \cap S = \varnothing$ since $\gamma([0, \frac{1}{2}]) \subset L_1 \in L_x\backslash L_x^c$ and $\gamma([\frac{1}{2}, 1]) \subset L_2 \in L_y\backslash L_y^c$. Therefore the complement of S in $\mathbb{R}^2$ is path-connected. ■

Proposition 44. *Any open connected $A \subset \mathbb{R}^n$ is path-connected.*

Proof. If $A = \varnothing$, then it is vacuously path connected. Otherwise, there exists $x_0 \in A$. Let P denote the set of points in A path connected to x_0. Since $x_0 \in P$, P is nonempty.

Consider any $x \in P$ and let γ_1 denote the path from x_0 to x. Since A is open, there exists $\delta > 0$ s.t. $B_\delta(x) \subset A$. For any $y \in B_\delta(x)$, let $\gamma_2(t) = (1-t)x + ty$. By the triangle inequality, $\gamma_2([0, 1]) \subset B_\delta(x) \subset A$. Then

$$\gamma(t) = \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

is a path in A from x_0 to y, hence $B_\delta(x) \subset P$ and P is open.

Now consider a point not path connected to x_0, that is $x \in A\backslash P$. Since A is open again there exists $\delta > 0$ s.t. $B_\delta(x) \subset A$. Suppose there were a path γ_1 from x_0 to $y \in B_\delta(x)$. Then compose this path again as before with a path from y to x. This is a path from x_0 to x, a contradiction. Hence y is also not path connected to x_0, and $B_\delta(x) \subset A\backslash P$, hence P is clopen. Since P is a nonempty clopen subset of A, $P = A$, and hence A is path connected since x_0 was arbitrary. ■

Proposition 45. *Let $\mathcal{S}, \mathcal{T}$ be two topologies on X with $\mathcal{S} \subset \mathcal{T}$. If X is compact under $\mathcal{T}$, it is compact under $\mathcal{S}$. However, if X is compact under $\mathcal{S}$, it is not necessarily compact under $\mathcal{T}$.*

Proof. ($\Longrightarrow$) Let $\bigcup_{\lambda \in \Lambda} U_\lambda$ be an open cover of X under $\mathcal{S}$, that is, with $U_\lambda \in \mathcal{S}$ for all λ. Since $\mathcal{S} \subset \mathcal{T}$, U_λ is also open in $\mathcal{T}$ for all λ, and hence the open cover of X under $\mathcal{S}$ is also an open cover under $\mathcal{T}$. But X is compact in $\mathcal{T}$, and thus there exists a finite subcover $\bigcup_{\lambda \in \Lambda'} U_\lambda$ with $\Lambda' \subset \Lambda$ finite. This finite subcover under $\mathcal{T}$ is also a finite subcover under $\mathcal{S}$, since all $U_\lambda \in \mathcal{S}$, and hence X is compact under $\mathcal{S}$.

($\not\Longleftarrow$) Consider any set X compact under $\mathcal{S}$. Let $\mathcal{T} = \mathcal{P}(X)$. We have $\mathcal{S} \subset \mathcal{T}$, but X cannot be compact under $\mathcal{T}$, since an open cover by singletons has no finite subcover. ■

Proposition 46. *If X is compact Hausdorff under both $\mathcal{S}$ and $\mathcal{T}$ with $\mathcal{S} \subset \mathcal{T}$, then $\mathcal{S} = \mathcal{T}$.*

Proof. Let $X_\mathcal{S}$ denote X under $\mathcal{S}$ and let $X_\mathcal{T}$ denote X under $\mathcal{T}$. Consider the identity map $i : X_\mathcal{T} \to X_\mathcal{S}$ given by $i(x) = x$. The preimage of any open set U under the map is itself. Because $\mathcal{T}$ is finer than $\mathcal{S}$, U must be open in the domain, and hence i is continuous. Since i is a bijection and the domain is compact and the range is Hausdorff, i must be a homeomorphism, and in particular, its inverse i^{-1} must be continuous. This implies $\mathcal{T} \subset \mathcal{S}$, and thus $\mathcal{S} = \mathcal{T}$. ■

Proposition 47. *Any topological space X with the cofinite topology is compact.*

Proof. If X is empty, it is vacuously compact. Otherwise, let $\bigcup_{\lambda \in \Lambda} U_\lambda$ be an open cover of X. Since X is nonempty, there exists a nonempty U_{λ_0} for some $\lambda_0 \in \Lambda$. U_{λ_0} nonempty and open in the cofinite topology implies $X \backslash U_{\lambda_0}$ is finite. For all $x \in X \backslash U_{\lambda_0}$, let U_x denote any U_λ in the open cover containing x. There must exist at least one such set for all x by the definition of cover. Then $U_{\lambda_0} \cup \bigcup_{x \in X \backslash U_{\lambda_0}} U_x$ is a finite subcover of X, hence X is compact. ■

Proposition 48. *Let the cocountable topology on $\mathbb{R}$ be the topology under which $U \subset \mathbb{R}$ is open if and only if either $U = \varnothing$ or $\mathbb{R} \backslash U$ is countable. Then $\mathbb{R}$ under the cocountable topology is not compact.*

Proof. Consider $\bigcup_{n \in \mathbb{N}} \mathbb{R} \backslash \mathbb{N} \cup \{n\}$. Each $\mathbb{R} \backslash \mathbb{N} \cup \{n_0\}$ with $n_0 \in \mathbb{N}$ is open since its complement, $\mathbb{N} \backslash \{n_0\}$, is countable. Additionally, for all $x \in \mathbb{R}$, if $x \in \mathbb{R} \backslash \mathbb{N}$, $x \in \mathbb{R} \backslash \mathbb{N} \cup \{0\}$. If $x = n_0$ for some $n_0 \in \mathbb{N}$, then $x \in \mathbb{R} \backslash \mathbb{N} \cup \{n_0\}$. Hence the union is an open cover of $\mathbb{R}$ under the cocountable topology. However, no proper subcollection of the open cover is a cover, since if any $\mathbb{R} \backslash \mathbb{N} \cup \{n_0\}$ is missing for some $n_0 \in \mathbb{N}$, then n_0 is missing from the cover. Since this open cover has no finite subcover, $\mathbb{R}$ is not compact under this topology. ■

Proposition 49. *Let $\{A_n | n \in \mathbb{N}\}$ be a countable family of compact, connected subsets of a Hausdorff space X such that $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$. Let $A = \bigcap_{n \in \mathbb{N}} A_n$. A is nonempty if and only if each A_n is nonempty.*

Proof. ($\Longrightarrow$) If A_{n_0} is empty for some $n_0 \in \mathbb{N}$, then $A = \bigcap_{n \in \mathbb{N}} A_n \subset A_{n_0} = \varnothing$ implies $A = \varnothing$.

($\Longleftarrow$) Suppose for a contradiction that A_n is nonempty for all $n \in \mathbb{N}$ but A is empty. Consider $U = \bigcup_{n\in\mathbb{N}\backslash\{0\}} A_0\backslash A_n$.

Since each A_n is compact and X is Hausdorff, each A_n is closed in X. Additionally, each $A_n \subset A_0$ for all $n > 0$ by induction, so they must be closed in the subspace topology on A_0, thus their complements $A_0\backslash A_n$ must be open in the subspace topology on A_0. A empty implies U must be a cover of A_0 because if $x \in A_0$ but $x \notin A_0\backslash A_n$ for all $n > 0 \in \mathbb{N}$, then $x \in A_n$ for all $n > 0$. But since $x \in A_0$ and $x \in A_n$ for all $n > 0$, $x \in A$, a contradiction to $A = \varnothing$. Thus U is an open cover of A_0.

A_0 compact implies U admits a finite subcover $\bigcup_{n\in\Lambda} A_0\backslash A_n$ with $\Lambda \subset \mathbb{N}\backslash\{0\}$ finite. Since each $A_n \supset A_{n+1}$, $A_0\backslash A_n \subset A_0\backslash A_{n+1}$. Since Λ is a nonempty, finite subset of $\mathbb{N}$, it contains a maximal element N. By induction, $A_0\backslash A_N \supset \bigcup_{n\in\Lambda} A_0\backslash A_n$. But this gives $A_0\backslash A_N \supset A_0$, which implies $A_N = \varnothing$ since $A_N \subset A_0$, a contradiction. Hence A cannot be empty. ∎

Proposition 50. *Let $\{A_n|n \in \mathbb{N}\}$ be a countable family of compact, connected subsets of a Hausdorff space X such that $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$. Let $A = \bigcap_{n\in\mathbb{N}} A_n$. A is compact.*

Proof. Since each A_n is compact and X is Hausdorff, each A_n is closed and hence A is closed, since it is an intersection of closed sets. Because A is a closed subset of A_0 compact, it is compact in the subspace topology on A_0, hence compact in X, since the topology induced on A by A_0 is the same as the topology induced on A by X. ∎

Proposition 51. *Let $\{A_n|n \in \mathbb{N}\}$ be a countable family of compact, connected subsets of a Hausdorff space X such that $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$. Let $A = \bigcap_{n\in\mathbb{N}} A_n$. A is connected.*

Proof. Suppose for a contradiction A is not connected. Then $A = C \cup D$ with C, D clopen, disjoint, and nonempty. Since C and D are closed in A compact and closed, C and D are compact in A, and hence X by the argument in the proof of **Proposition 50**. X Hausdorff implies there exist disjoint, open $U, V \subset X$ containing C and D respectively.

Since $A \subset C \cup D \subset U \cup V$, $A\backslash(U \cup V) = \varnothing$. Rewriting, $A\backslash(U \cup V) = (\bigcap_{n\in\mathbb{N}} A_n) \backslash (U \cup V) = \bigcap_{n\in\mathbb{N}} (A_n\backslash(U \cup V))$. Since $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$, we have $A_n\backslash(U \cup V) \supset A_{n+1}\backslash(U \cup V)$ for all $n \in \mathbb{N}$. Additionally, $A_n\backslash(U \cup V) = A_n \cap (X\backslash(U \cup V))$ for all $n \in \mathbb{N}$, an intersection of closed sets, hence closed. Thus $A_n\backslash(U \cup V)$ are closed subsets of A_n compact, hence also compact for all $n \in \mathbb{N}$.

Since we have a family of nested, compact subsets with an empty intersection, by **Proposition 49**, it must be that $A_{n_0}\backslash(U \cup V) = \varnothing$ for some $n_0 \in \mathbb{N}$, and thus $A_{n_0} \subset U \cup V$. Note $A \cap C \neq \varnothing$ implies $A_{n_0} \cap U \neq \varnothing$ and $A \cap D \neq \varnothing$ implies $A_{n_0} \cap V \neq \varnothing$. Since $A_{n_0} \cap U$ and $A_{n_0} \cap V$ are nonempty, disjoint clopen subsets of A_{n_0} that cover A_{n_0}, A_{n_0} cannot be connected, a contradiction. ∎

Proposition 52. *Let X, Y be topological spaces with Y compact. Then the projection $\pi : X \times Y \to X$ is closed.*

Proof. Suppose for a contradiction $C \subset X \times Y$ is closed but $\pi(C)$ is not closed. Then $\pi(C)$ has a limit point x_0 outside of itself. $\pi^{-1}(\{x_0\}) = \{x_0\} \times Y$ must be contained in the complement of C closed, thus for every $x = (x_0, y) \in \pi^{-1}(\{x_0\})$, we have a basic neighborhood $U(y) \times V(y)$ of x contained in the complement of C. $\bigcup_{y \in Y} V(y)$ is an open cover of Y and thus admits a finite subcover since Y is compact. Letting Λ index the finite subcover, we consider $U = \bigcap_{y \in \Lambda} U(y)$, an open set since Λ is finite. C is nonempty since $\pi(C)$ is not closed, so consider $(c_1, c_2) \in C$. If U intersects[1] $\pi(C)$ at some point x_1, we have $(x_1, c_2) \in U(c_2) \times V(c_2)$ and $(x_1, c_2) \in C$. But the $U(y) \times V(y)$ were chosen to miss C, so U misses $\pi(C)$, yet there cannot be an open neighborhood of x_0 missing $\pi(C)$ since $x_0 \in \overline{\pi(C)}$. ∎

Proposition 53. *Every closed subset of a countably compact space is countably compact.*

Proof. The proof is exactly similar to the proof of closed subsets of compact spaces being compact, replacing each instance of 'open cover' with 'countable open cover' in the proof. ∎

Proposition 54. *Let A be a subset of a T_1 space X. If x is a limit point of A, then every open neighborhood of x contains infinitely many points of A.*

Proof. Let U be an open neighborhood of x and suppose $U \cap A$ is finite. Then $(U \cap A)\backslash\{x\}$ is closed, since it is finite and X is T_1. Then $X\backslash((U \cap A)\backslash\{x\}) = \{x\} \cup X\backslash(U \cap A)$ is open, and $(\{x\} \cup X\backslash(U \cap A)) \cap U$ is an open neighborhood of x not intersecting A at any point other than x, but x is a limit point of A. ∎

Proposition 55. *A T_1 space X is countably compact if and only if it is limit point compact.*

Proof. ($\Longrightarrow$) Suppose there exists an infinite subset B of X with no limit points. Let A be a countable subset of B, which again cannot have any limit points. By **Proposition 34**, the subspace topology on A is discrete. Hence $\bigcup_{a \in A}\{a\}$ is a countable open cover of A with no finite subcover, so A cannot be countably compact. Since A has no limit points, it contains all of its limit points, and thus is closed. A closed but not countably compact implies X is not countably compact by **Proposition 53**.

($\Longleftarrow$) Let $\bigcup_{i=0}^{\infty} U_i$ be a countable open cover of X. If no finite subcollection covers X, let x_n be a point not in $U_0 \cup \ldots \cup U_{n-1}$ and let U_n in the cover contain x_n. Let $A = \bigcup_{i \in \mathbb{N}}\{x_i\}$. Note that for all $i \in \mathbb{N}$, $U_i \cap A$ must be finite, since U_i cannot contain any x_n for $n > i$. Since the U_i cover X, all points in X have an open neighborhood which intersects A finitely many times, so no point can be a limit point of A by **Proposition 54**. ∎

Proposition 56. *Let X be a metric space. For nonempty $A, B \subset X$, define $d(A, B) = \inf\{d(x, y) | x \in A, y \in B\}$. Also define for all $x \in X$, $d(x, A) = d(\{x\}, A)$. For $x \in X$, $d(x, A) = 0$ if and only if $x \in \overline{A}$.*

[1] If $U \cap V = \varnothing$, then U *misses* V. If $U \cap V \neq \varnothing$, then U *intersects* V.

Proof. ($\Longrightarrow$) By the approximation property of the infimum, for all $\delta > 0$, there exists $a \in A$ such that $0 \leqslant d(x, a) < \delta$. Then for all $\delta > 0$, $B_\delta(x) \cap A$ is nonempty, so $x \in \overline{A}$.

($\Longleftarrow$) $x \in \overline{A}$ implies for all $\delta > 0$, $B_\delta(x) \cap A$ is nonempty. Then for all $\delta > 0$ there exists $a \in A$ such that $d(x, a) < \delta$, so $d(x, A) = 0$ since it is a nonnegative number less than all $\delta > 0$. ∎

Proposition 57. *Let X be a metric space. If A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$.*

Proof. Define $d_x : A \to \mathbb{R}$ to be $d(x, a)$ for all $a \in A$. d_x is continuous since the inverse image of any basic open interval (a, b) is $(B_b(x)\backslash C_a(x)) \cap A$, open in the subspace topology on A. Since d is a continuous function from a compact set to $\mathbb{R}$, the infimum is in the image of d by the extreme value theorem, but the infimum is $d(x, A)$. ∎

Proposition 58. *Let X be a metric space. Define $B_\delta(A) = \{x \in X | d(x, A) < \delta\}$. Then $B_\delta(A) = \bigcup_{a \in A} B_\delta(a)$.*

Proof. ($\subset$) $x \in B_\delta(A)$ implies $d(x, A) < \delta$, which by the approximation property of the infimum means there exists $a_0 \in A$ such that $d(x, A) \leqslant d(x, a_0) < \delta$ so $x \in B_\delta(a_0)$ and hence $x \in \bigcup_{a \in A} B_\delta(a)$.

($\supset$) $x \in \bigcup_{a \in A} B_\delta(a)$ implies $x \in B_\delta(a_0)$ for some $a_0 \in A$, and $d(x, a_0) < \delta$ implies $d(x, A) < \delta$ and hence $x \in B_\delta(A)$. ∎

Proposition 59. *Let X be a metric space. Suppose A is compact and $U \subset X$ is an open set containing A. Then there exists $\delta > 0$ such that $B_\delta(A) \subset U$.*

Proof. If $U = X$, then any $\delta > 0$ will suffice. Otherwise, let $d_{X\backslash U} : A \to \mathbb{R}$ be given by $d(a, X\backslash U)$ for all $a \in A$. For two points a_1, a_2, we have $d(a_1, X\backslash U) = \inf_{x \in X\backslash U} d(a_1, x) \leqslant \inf_{x \in X\backslash U} d(a_1, a_2) + d(a_2, x) = d(a_1, a_2) + d(a_2, X\backslash U)$. Therefore $|d(a_1, X\backslash U) - d(a_2, X\backslash U)| \leqslant d(a_1, a_2)$. Hence for all $\epsilon > 0$, $d(a_1, a_2) < \epsilon$ implies $|d_{X\backslash U}(a_1) - d_{X\backslash U}(a_2)| < \epsilon$, thus $d_{X\backslash U}$ is continuous.

Since $d_{X\backslash U}$ is a continuous function with a compact domain, it attains its minimum for some $a_m \in A$. Since A is a compact subset of a Hausdorff space, it is closed, and hence $d_{X\backslash U}(a_m) = \delta > 0$ since $X\backslash U$ cannot contain points in the closure of A. Hence $\bigcup_{a \in A} B_{d_{X\backslash U}(a_m)}(a) = B_{d_{X\backslash U}(a_m)}(A) \subset U$. ∎

Proposition 60. *Let X be a metric space. Let $A = \{(x, y) | y \leqslant 0\}$ be the lower halfplane of $\mathbb{R}^2$. Let $U = \{(x, y) | (x, y) < (x, e^x)\}$. Then there exists no $\delta > 0$ such that $B_\delta(D) \subset U$.*

Proof. A is closed since its complement $\mathbb{R} \times (0, \infty)$ is open. Suppose there exists $\delta > 0$ such that $B_\delta(A) = \bigcup_{a \in A} B_\delta(a)$ does not intersect $X\backslash U$. Then for all $a = (a_1, 0) \in A$, $(a_1, e^{a_1}) \notin B_\delta(a_1, 0)$. But $d((a_1, e^{a_1}), (a_1, 0)) = e^{a_1}$, which is less than δ for $a_1 < \ln \delta$. ∎

Proposition 61. *Let X be a metric space. Let $f : X \to X$ be an isometry and X be compact. f is surjective.*

Proof. f is continuous and injective as was shown on the midterm. Suppose for a contradiction $a \notin f(X)$ for some $a \in X$. Since X is compact and f is continuous, $f(X)$ is compact. Since X is a metric space, it is Hausdorff, thus $f(X)$ is closed. Hence $a \in X \backslash f(X)$ implies there exists $\epsilon > 0$ such that $B_\epsilon(a)$ misses $f(X)$. Define a sequence inductively by $x_0 = a$ and $x_{n+1} = f(x_n)$. Since $B_\epsilon(a)$ misses $f(X)$, $d(x_0, x_i) \geqslant \epsilon$ for all $i > 0$. Also, $d(x_i, x_{i+1}) = d(f(x_i), f(x_{i+1})) = d(x_{i+1}, x_{i+2})$. For any $m < n$, $d(x_0, x_{n-m}) \geqslant \epsilon$ implies $d(f^m(x_0), f^m(x_{n-m})) = d(x_m, x_n) \geqslant \epsilon$ by induction.

If X is finite, f is surjective since it is injective, so assume X is infinite. Additionally, $x_n \neq x_m$ for $m \neq n$, since otherwise if $m \neq n$ and $x_n = x_m$, then $x_{n-1} = x_{m-1}$ since f is injective. But then by induction $x_0 = x_i$ for some $i > 0$, which cannot be since $x_0 = a \notin f(X)$. Therefore the sequence must be infinite. X compact implies this sequence has a limit point x. By **Proposition 54**, $B_{\epsilon/2}(x)$ intersects the sequence at infinitely many points. But if x_n and x_m are contained in $B_{\epsilon/2}(x)$ for $n \neq m$, then $d(x_n, x_m) \leqslant d(x_n, x) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon$, a contradiction. ∎

Proposition 62. *Let X be a metric space. Let $f : X \to X$ be an isometry and X be compact. f is a homeomorphism.*

Proof. Since f is a continuous bijection from a compact space to a Hausdorff space, it is a homemorphism. ∎

Proposition 63. *Let X be a metric space. Let $f : X \to X$ be a contraction and X compact. f is continuous.*

Proof. For all $\epsilon > 0$, $d(x, y) < \epsilon$ implies $d(f(x), f(y)) \leqslant cd(x, y) < c\epsilon < \epsilon$. ∎

Proposition 64. *Let X be a metric space. Let $f : X \to X$ be a contraction and X compact. If X is nonempty, then f has exactly one fixed point.*

Proof. X nonempty implies $f^n(X)$ is nonempty for all $n \in \mathbb{N}$ by induction[2]. f continuous and X compact implies $f^n(X)$ is compact for all $n \in \mathbb{N}$. $X \supset f(X)$ also implies $f^n(X) \supset f^{n+1}(X)$ by applying f to both sets and using induction. Therefore we have a countable family of nested compact sets which are all nonempty and subsets of a Hausdorff space. By **Proposition 49** and **Proposition 50**, $A = \bigcap_{n \in \mathbb{N}} f^n(X)$ is nonempty and compact.

Let x be any point in A. Then for all $n \in \mathbb{N}$, there exists x_n such that $x = f^n(x_n)$. Hence $d(x, f(x)) = d(f^n(x_n), f^{n+1}(x_n)) \leqslant c^n d(x_n, f(x_n))$ by induction. A compact in a metric space implies there exists $\delta > 0$, $x \in X$ such that $B_\delta(x) \supset A$, so $d(x_n, f(x_n)) < \delta$. This gives $d(x, f(x)) \leqslant c^n \delta$ for all $n \in \mathbb{N}$, so $d(x, f(x)) = 0$. Hence $x = f(x)$. Thus every point in A is a fixed point of f and there must be at least one such point.

Let x and y be two fixed points of f. Then $d(f(x), f(y)) \leqslant cd(x, y)$ so $d(x, y) \leqslant cd(x, y)$ and $(1 - c)d(x, y) \leqslant 0$. Since $c < 1$ and is nonnegative, $d(x, y) = 0$ and hence $x = y$. Hence a fixed point of f exists and is unique. ∎

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[2] $f^n(X)$ denotes n-fold composition of f

Proposition 65. *$\bigcup_{n\in\mathbb{N}}(\frac{1}{2^{n+2}}, \frac{1}{2^n})$ is an open cover of $(0,1)$ where the Lebesgue Number Lemma fails.*

Proof. We have shown already in class that this is an open cover. Suppose there exists $\delta > 0$ such that every subset $A \subset B_\delta(x)$ for some $x \in (0,1)$ is contained in some set of the open cover. For all $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\left|\frac{1}{2^{n+2}} - \frac{1}{2^n}\right| < \delta$. Then $B_\delta(\frac{1}{2^{n+1}} + \frac{1}{2^{n+3}})$ is not contained in $(\frac{1}{2^{n+2}}, \frac{1}{2^n})$. But since it contains $\frac{1}{2^{n+1}}$ and $(\frac{1}{2^{n+2}}, \frac{1}{2^n})$ is the only set in the cover containing $\frac{1}{2^{n+1}}$, it cannot be contained in any other set of the open cover, a contradiction. ■

Proposition 66. *The 1-point compactification $\widehat{\mathbb{R}}$ of $\mathbb{R}$ is homemorphic to the unit circle S^1.*

Proof. Let $f : \mathbb{R} \to S^1\backslash\{(0,1)\}$ be given by $f(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$. $t^2 + 1 > 0$ for all $t \in \mathbb{R}$, so the rational coordinate functions are continuous, thus f is continuous by **Proposition 21**.

Let $g : S^1\backslash\{(0,1)\} \to \mathbb{R}$ be $g(u,v) = \frac{u}{1-v}$. $(1-v) \neq 0$ for all $(u,v) \in S^1\backslash\{(0,1)\}$, so g has continuous partial derivatives, and thus is diffentiable, hence continuous.

$(g \circ f)(t) = \left(\frac{2t}{t^2+1}\right) / \left(1 - \frac{t^2-1}{t^2+1}\right) = \left(\frac{2t}{t^2+1}\right) / \left(\frac{2}{t^2+1}\right) = t.$

$(f \circ g)(u,v) = \left(\left(\frac{2u}{1-v}\right) / \left(1 + \frac{u^2}{(1-v)^2}\right), \left(\frac{u^2}{(1-v)^2} - 1\right) / \left(\frac{u^2}{(1-v)^2} + 1\right)\right)$

$= \left(\frac{2u(1-v)}{u^2+v^2+1-2v}, \frac{u^2-(1-v)^2}{u^2+(1-v)^2}\right) = \left(\frac{2u(1-v)}{2-2v}, \frac{u^2+v^2-1+2v-2v^2}{2-2v}\right) = (u,v)$

Since f has a double sided inverse, it is a bijection. It is also continuous, and its inverse is continuous, so it is a homemorphism. $\mathbb{R} \simeq S^1\backslash\{(0,1)\}$ implies $\widehat{\mathbb{R}} \simeq S^1$ since S^1 is compact and Hausdorff and $\mathbb{R}$ is locally compact and Hausdorff and noncompact. ■

Proposition 67. *D dense in X and U nonempty, open in X implies $\overline{D \cap U} = \overline{U}$.*

Proof. $(\subset)$ $D \cap U \subset U$ implies $\overline{D \cap U} \subset \overline{U}$. $(\supset)$ Let V be any open neighborhood of $x \in \overline{U}$. Then $V \cap U$ is nonempty by definition of closure and open since it is the intersection of two open sets. D dense in X implies $D \cap (V \cap U)$ is nonempty, so $V \cap (D \cap U)$ is nonempty, hence $x \in \overline{D \cap U}$. ■

Proposition 68. *No compact subset of $\mathbb{Q}$ contains $(a,b) \cap \mathbb{Q}$ for any $a < b \in \mathbb{R}$.*

Proof. Suppose for a contradiction A compact in $\mathbb{Q}$ and $\mathbb{Q} \cap (a,b) \subset A$ for some $a < b \in \mathbb{R}$. A compact in $\mathbb{Q}$ implies A compact in $\mathbb{R}$ since the inclusion map is continuous. A compact in $\mathbb{R}$ Hausdorff implies $\overline{A} = A$. $\mathbb{Q}$ dense in $\mathbb{R}$ implies $\overline{\mathbb{Q} \cap (a,b)} = [a,b]$ by **Proposition 67**. Hence $\mathbb{Q} \cap (a,b) \subset A$ implies $[a,b] \subset A$, a contradiction since $[a,b] \not\subset \mathbb{Q}$ but $A \subset \mathbb{Q}$. ■

Proposition 69. *$\mathbb{Q}$ is not locally compact.*

Proof. Let $U \cap \mathbb{Q}$ be an open neighborhood of $q \in \mathbb{Q}$. U open in the metric topology on $\mathbb{R}$ implies $U \cap \mathbb{Q} = \bigcup_{\lambda\in\Lambda}(a_\lambda, b_\lambda) \cap \mathbb{Q}$. By **Proposition 68**, any compact set in $\mathbb{Q}$ does not contain any $(a_\lambda, b_\lambda) \cap \mathbb{Q}$ for any λ. Hence no compact set contains $U \cap \mathbb{Q}$, so q does not have a compact neighborhood. ■

Proposition 70. *$f : \mathbb{R} \to \mathbb{R}$ continuous, then*

$$(1) \lim_{x\to\infty} |f(x)| = \infty \implies (2) \lim_{x\to\infty} f(x) = \pm\infty$$

$$(3) \lim_{x\to-\infty} |f(x)| = \infty \implies (4) \lim_{x\to-\infty} f(x) = \pm\infty$$

Proof. If not (2), then $\exists M, M' \in \mathbb{R}, \forall N \in \mathbb{R}, \exists x, x' \in \mathbb{R}, (x > N \wedge f(x) < M \wedge x' > N \wedge f(x') > M')$. Let $M'' = \min(M-1, M')$. Then $\forall N \in \mathbb{R}, \exists x \in \mathbb{R}, (x > N \wedge f(x) > M'')$ since $M'' \leqslant M'$. By the Intermediate Value Theorem, $\forall N \in \mathbb{R}, \exists x'' \in \mathbb{R}, (x'' > N \wedge (M'' < f(x'') < M))$. $M'' < f(x'') < M$ implies $|f(x'')| < \max(|M|, |M''|)$, since $-f(x'') < -M'' \leqslant |M''|$ and $f(x) < M \leqslant |M|$. Let $\widetilde{M} = \max(|M|, |M''|)$. Then $\forall N \in \mathbb{R}, \exists x \in \mathbb{R}, (x > N \wedge |f(x)| < \widetilde{M})$, so not (1). The proof of not (4) implies not (3) is similar. ∎

Proposition 71. *$f : \mathbb{R} \to \mathbb{R}$ continuous is proper if and only if* (2) *and* (4)*.*

Proof. ($\implies$) If not (2), then $\exists M \in \mathbb{R}, \forall N \in \mathbb{R}, \exists x \in \mathbb{R}, (x > N \wedge |f(x)| < M)$ by **Proposition 70**. Then $\forall N \in \mathbb{R}, \exists x \in \mathbb{R}, (x > N \wedge x \in f^{-1}([-M, M]))$, so $f^{-1}([-M, M])$ is unbounded, hence not compact by Heine-Borel. Since $[-M, M]$ is compact but $f^{-1}([-M, M])$ is not, f is not proper. The proof of not (4) implies f is not proper is similar.

($\impliedby$) Since f is continuous, the inverse image of closed sets is closed. Let $C \subset \mathbb{R}$ be bounded, thus contained in $B_\delta(0)$ for some $\delta > 0$. (2) implies $\exists N \in \mathbb{R}, \forall x \in \mathbb{R}, (x > N \implies f(x) > \delta)$. If $f^{-1}(C)$ unbounded above, then $\forall M \in \mathbb{R}, \exists x \in \mathbb{R}, (x > M \wedge x \in f^{-1}(C))$. Let $M = N$. Then $\exists x \in \mathbb{R}, (x > N \wedge x \in f^{-1}(C) \wedge f(x) > \delta)$, but $f(C) \subset B_\delta(0)$, a contradiction, so $f^{-1}(C)$ is bounded above. A similar argument shows $f^{-1}(C)$ is bounded below using (4). Since the inverse image of closed and bounded sets is closed and bounded, by Heine-Borel, the inverse image of compact sets is compact, so f is proper. ∎

Proposition 72. *Nonconstant polynomial functions $p : \mathbb{R} \to \mathbb{R}$ are proper.*

Proof. From analysis we know nonconstant polynomial functions from $\mathbb{R} \to \mathbb{R}$ are continuous and satisfy (2) and (4), hence by **Proposition 71**, are proper. ∎

Proposition 73. *For X, Y Hausdorff, continuous $f : X \to Y$ is proper if and only if $\widehat{f} : \widehat{X} \to \widehat{Y}$ given by*

$$\widehat{f}(x) = \begin{cases} f(x) & x \in X \\ \infty & x = \infty \end{cases}$$

is continuous.

Proof. ($\implies$) Let U be open in $\widehat{Y}$. If $\infty \notin U$, then U is open in Y and $\infty \notin \widehat{f}^{-1}(U)$. Hence $\widehat{f}^{-1}(U) = f^{-1}(U)$, open in X since the inverse image of an open set in Y under f is an open set in X. But open in X also implies open in $\widehat{X}$ by definition of $\widehat{X}$.

If $\infty \in U$, then $U = Y \backslash C \cup \{\infty\}$ for some C compact in Y. We have $\widehat{f}^{-1}(U) = f^{-1}(Y \backslash C) \cup \{\infty\} = f^{-1}(Y) \backslash f^{-1}(C) \cup \{\infty\} = X \backslash f^{-1}(C) \cup \{\infty\}$. Since f is proper,

C compact in Y implies $f^{-1}(C)$ compact in X, hence $X\backslash f^{-1}(C) \cup \{\infty\}$ is open in $\widehat{X}$.

($\Longleftarrow$) Let C be any compact set in Y. Then $Y\backslash C \cup \{\infty\}$ is open in $\widehat{Y}$. $\widehat{f}$ continuous implies $\widehat{f}^{-1}(Y\backslash C \cup \{\infty\}) = f^{-1}(Y\backslash C) \cup \{\infty\} = X\backslash f^{-1}(C) \cup \{\infty\}$ open in $\widehat{X}$, hence $f^{-1}(C)$ must be compact in X, and thus f is proper. ∎

Proposition 74. *If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are continuous, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is continuous.*

Proof. Let U be any open set in $Y_1 \times Y_2$. By the definition of the product topology, $U = \bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda$ for U_λ open in Y_1, V_λ open in Y_2. We have $(f_1 \times f_2)^{-1}(U) = (f_1 \times f_2)^{-1}(\bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda) = \bigcup_{\lambda \in \Lambda} (f_1 \times f_2)^{-1}(U_\lambda \times V_\lambda) = \bigcup_{\lambda \in \Lambda} f_1^{-1}(U_\lambda) \times f_2^{-1}(V_\lambda)$. Since f_1 and f_2 are continuous, $f_1^{-1}(U_\lambda)$ is open in X_1, $f_2^{-1}(V_\lambda)$ is open in X_2, so $f_1^{-1}(U_\lambda) \times f_2^{-1}(V_\lambda)$ is open in $X_1 \times X_2$ for all $\lambda \in \Lambda$. Hence $(f_1 \times f_2)^{-1}(U)$ is open in $X_1 \times X_2$ since it is a union of open sets. ∎

Proposition 75. *Suppose $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are continuous, X_1 and X_2 are nonempty, and Y_1 and Y_2 are Hausdorff. Then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is proper if and only if f_1 and f_2 are proper.*

Proof. ($\Longrightarrow$) Let $C_1 \subset Y_1$ be compact. Since X_2 is nonempty, choose $x_2 \in X_2$. $C_1 \times \{f_2(x_2)\}$ is compact in $Y_1 \times Y_2$ since the product of two compact sets is compact, so $(f_1 \times f_2)^{-1}(C_1 \times \{f_2(x_2)\}) = f_1^{-1}(C_1) \times f_2^{-1}(\{f_2(x_2)\})$ is compact. Let $\pi_1 : X_1 \times X_2 \to X_1$ be the natural projection. $\pi_1(f_1^{-1}(C_1) \times f_2^{-1}(\{f_2(x_2)\})) = f_1^{-1}(C_1)$ since $f_2^{-1}\{f_2(x_2)\}$ is nonempty. Since π_1 is continuous, $f_1^{-1}(C_1)$ is compact, thus f_1 is proper. f_2 is proper by a similar argument.

($\Longleftarrow$) Let K be any compact subset of $Y_1 \times Y_2$. Again, $\pi_1(K)$ and $\pi_2(K)$ are compact and $\pi_1(K) \times \pi_2(K)$ is compact. Hence $(f_1 \times f_2)^{-1}(\pi_1(K) \times \pi_2(K)) = f_1^{-1}(\pi_1(K)) \times f_2^{-1}(\pi_2(K))$ is compact since f_1 and f_2 are proper. Since Y_1 and Y_2 are Hausdorff, $Y_1 \times Y_2$ is Hausdorff, hence K is closed. By **Proposition 74**, $f_1 \times f_2$ is continuous, so $(f_1 \times f_2)^{-1}(K)$ is closed. But $(f_1 \times f_2)^{-1}(K) \subset (f_1 \times f_2)^{-1}(\pi_1(K) \times \pi_2(K))$ compact, so it is compact. ∎

Proposition 76. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, $g \circ f$ is proper, and Y is Hausdorff, then f is proper.*

Proof. Let C be a compact set in Y. Since g is continuous, $g(C)$ is compact in Z. $g \circ f$ is proper thus $(g \circ f)^{-1}(g(C))$ is compact in X. Y is Hausdorff hence C is closed. f is continuous implies $f^{-1}(C)$ is closed. $C \subset g^{-1}(g(C))$ implies $f^{-1}(C)$ is a closed subset of $(g \circ f)^{-1}(g(C))$ compact, so it is compact. ∎

Proposition 77. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, $g \circ f$ is proper, and f is surjective, then g is proper.*

Proof. Let C be a compact set in Z. Then $(g \circ f)^{-1}(C)$ is a compact set in X since $g \circ f$ is proper. Since f is continuous, $f((g \circ f)^{-1}(C))$ is compact in Y. But f is surjective, thus $f((g \circ f)^{-1}(C)) = g^{-1}(C)$. ∎

Proposition 78. *If $f_0, f_1 : X \to Y$ are homotopic, $g_0, g_1 : Y \to Z$ are homotopic, then $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic.*

Proof. Let F be the homotopy between f_0 and f_1 and let G be the homotopy between g_0 and g_1. Let $H = G \circ (F \times \mathrm{id}_{[0,1]})$. H is continuous since it is a composition of continuous functions. We have $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$ and $H(x,1) = G(F(x,1),1) = G(f_1(x),1) = g_1 \circ f_1(x)$, hence $g_0 \circ f_0 \sim g_1 \circ f_1$. ∎

Proposition 79. *All intervals in $\mathbb{R}$ are contractible.*

Proof. Intervals are convex and any two continuous maps onto a convex set are homotopic by the straight line homotopy, hence the identity map is homotopic to any constant map. ∎

Proposition 80. *Any contractible X is path-connected and has $\pi_1(X,x) \cong 1$ for all $x \in X$.*

Proof. Let F be the homotopy between the identity map and the constant map $f(x) = x_0$ for some contraction point $x_0 \in X$. For all $x \in X$, let $\gamma_x = F|_{\{x\}\times I}$. γ_x is continuous since restrictions of continuous functions are continuous. $\gamma_x(0) = F(x,0) = \mathrm{id}(x) = x$ and $\gamma_x(1) = F(x,1) = f(x) = x_0$, hence it is a well-defined path from x to x_0. We can thus construct a path between any two $x_1, x_2 \in X$ using $\gamma_{x_1} * \widehat{\gamma_{x_2}}$ [3].

Let γ be any loop with basepoint x_0. Note $F(\gamma(s),t)$ gives a homotopy between $\gamma(s)$ and the constant loop $e_{x_0}(s) = x_0$. However, the basepoint need not be fixed, so this is not necessarily a path-homotopy.

Consider the image of the basepoint, $\gamma_{x_0}(s) = F(x_0,s)$, which is a loop based at x_0 since x_0 is the contraction point. We can prepend and append this path to the homotopy to fix the basepoint in place. Let $\gamma^t_{x_0}(s) = \gamma_{x_0}(ts)$ and $\gamma^t(s) = F(\gamma(s),t)$. Let $H(s,t) = \left(\widehat{\gamma^t_{x_0}} * \gamma^t * \gamma^t_{x_0}\right)(s)$. This is well defined since $\gamma^t_{x_0}(1) = F(x_0, 1 \times t) = F(x_0,t) = F(\gamma(0),t) = \gamma^t(0) = \gamma^t(1) = \widehat{\gamma^t_{x_0}}(0)$. The basepoint is fixed since $H(0,t) = \gamma^t_{x_0}(0) = x_0$ and $H(1,t) = \widehat{\gamma^t_{x_0}}(1) = x_0$. Hence H is a path-homotopy from $H(s,0) = \left(\widehat{\gamma^0_{x_0}} * \gamma^0 * \gamma^0_{x_0}\right)(s) = \left(\widehat{e_{x_0}} * F(\gamma,0) * e_{x_0}\right)(s) = \left(e_{x_0} * \gamma * e_{x_0}\right)(s)$ and $H(s,1) = \left(\widehat{\gamma^1_{x_0}} * \gamma^1 * \gamma^1_{x_0}\right)(s) = \left(\widehat{\gamma_{x_0}} * e_{x_0} * \gamma_{x_0}\right)(s)$. Finally, $\gamma \sim_p (e_{x_0} * \gamma * e_{x_0})$ and $e_{x_0} \sim_p (\widehat{\gamma_{x_0}} * e_{x_0} * \gamma_{x_0})$, hence $\gamma \sim_p e_{x_0}$, so $\pi_1(X,x_0) \cong 1$. Since X is path-connected and paths induce isomorphisms on fundamental groups with different bases, we have $\pi_1(X,x) \cong 1$ for all $x \in X$. ∎

Proposition 81. *If Y is contractible, then any two continuous $f_0, f_1 : X \to Y$ are homotopic.*

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[3] A wide hat over a path denotes the inverse path. That is, $\widehat{\gamma(s)} = \gamma(1-s)$

Proof. Let F be the homotopy from the identity map to the constant map $f(x) = x_0$ for some contraction point $x_0 \in X$. Let

$$H(x,t) = \begin{cases} F(f_0(x), 2t) & t \in [0, \frac{1}{2}] \\ F(f_1(x), 2-2t) & t \in [\frac{1}{2}, 1] \end{cases}$$

H is well defined since $F(f_0(x), 1) = F(f_1(x), 1) = x_0$ and it is continuous by the gluing lemma. It is a homotopy since $H(x, 0) = F(f_0(x), 0) = f_0(x)$ and $H(x, 1) = F(f_1(x), 0) = f_1(x)$. ∎

Proposition 82. *If X is contractible and Y is path-connected, then any two continuous $g_0, g_1 : X \to Y$ are homotopic.*

Proof. Let F be the homotopy from the identity on X to the constant function $f(x) = x_0$ for some contraction point $x_0 \in X$. Let $H(x,t) = g_0(F(x,t))$ and $G(x,t) = g_1(F(x,t))$. H and G are continuous since they are compositions of continuous functions and H is a homotopy from $H(x, 0) = g_0(F(x, 0)) = g_0(\mathrm{id}(x)) = g_0(x)$ to $H(x, 1) = g_0(F(x, 1)) = g_0(x_0)$. Similarly, G is a homotopy from $g_1(x)$ to $g_1(x_0)$. Let γ be a path from $g_0(x_0)$ to $g_1(x_0)$. Then $K(x,t) = \gamma(t)$ is a homotopy from $g_0(x_0)$ to $g_1(x_0)$. $g_0(x) \sim g_0(x_0) \sim g_1(x_0) \sim g_1(x)$, hence $g_0(x) \sim g_1(x)$. ∎

Proposition 83. *Let $S = \{(0, y) \mid y \in [0, 1]\} \cup \{(x, 0) \mid x \in [0, 1]\}$. $S \subset \mathbb{R}^2$ is star-convex but not convex.*

Proof. Let $x = (0, 0)$. Consider any $y = (x_1, y_1) \in S$. If $x_1 = 0$, then $\overline{xy} = (0, ty_1) \subset S$ for $t \in [0, 1]$ since $ty \in [0, 1]$ if both $t, y \in [0, 1]$. The argument is similar if $y_1 = 0$, so S is star-convex.

S is not convex since for $x = (1, 0)$ and $y = (0, 1)$, the line segment $\overline{xy} = (t, 0) + (0, 1-t)$ is not a subset of S since $(\frac{1}{2}, \frac{1}{2}) \notin S$. ∎

Proposition 84. *Let $f(s) = (\cos \pi s, \sin \pi s)$. Let $T = \{f(s) \mid s \in [0, 1]\}$. T is contractible but not star-convex.*

Proof. Let $F(s,t) = (1-t)s$. Let $G = f \circ F$. G is continuous since it is a composition of continuous functions. G is a contraction of T since $G(s, 0) = (\cos \pi s, \sin \pi s)$ and $G(s, 1) = (\cos 0, \sin 0) = (1, 0)$.

For any $p = (x_1, y_1) \in T$, we have $x_1^2 + y_1^2 = 1$ since $\cos^2 x + \sin^2 x = 1$ for all $x \in \mathbb{R}$. Let $y = mx + b$ be any line. This line intersects T at most twice since $x^2 + (mx + b)^2 = 1$ has at most two roots. Hence, T is not star-convex since any $\overline{xy}$ can contain at most two points of T but $\overline{xy}$ contains infinitely many points if $x \neq y$. ∎

Proposition 85. *Any star-convex set S is contractible hence has trivial fundamental group.*

Proof. Let $x_0 \in S$ be such that $\overline{x_0 y} \subset S$ for all $y \in S$. Let $F(x,t) = (1-t)x + tx_0$. F is continuous as we've shown in class. Since $\overline{x_0 x} \subset S$ for all x, $F(S,t) \subset S$, so F is a valid homotopy. Since $F(x, 0) = x$ and $F(x, 1) = x_0$, it is a contraction. ∎

Proposition 86. *Let $S \subset X$ and f be a retraction $X \to S$. Then for any $x_0 \in S$, the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(S, x_0)$ is surjective.*

Proof. Consider any $g \in \pi_1(S, x_0)$. Choose any $\gamma : I \to S$ as a representative of g. Let $i : S \to X$ be the natural inclusion map. Then $f_*(i_*([\gamma])) = f_*([i \circ \gamma]) = [f \circ i \circ \gamma] = [\mathrm{id}_S \circ \gamma] = [\gamma]$, hence f_* is surjective. ∎

Proposition 87. *For any two topological spaces X and Y, there is a natural isomorphism $\pi_1(X \times Y, x \times y) \cong \pi_1(X, x) \times \pi_1(Y, y)$.*

Proof. Let $\psi : \pi_1(X \times Y, x \times y) \to \pi_1(X, x) \times \pi_1(Y, y)$ be given by $\psi([\gamma]) = (\Pi_{1*}([\gamma]), \Pi_{2*}([\gamma]))$, where Π_1 and Π_2 denote the natural projections onto the first and second coordinates.

Consider any $g \in \pi_1(X \times Y, x \times y)$ and let $[\gamma_1] = [\gamma_2] = g$ for some representatives γ_1, γ_2. Then there exists $F(s,t)$ continuous such that $F(s,0) = \gamma_1(s), F(s,1) = \gamma_2(s), F(0,t) = F(1,t) = x \times y$. We have $\psi([\gamma_1]) = ([\Pi_1 \circ \gamma_1], [\Pi_2 \circ \gamma_1])$ and $\psi([\gamma_2]) = ([\Pi_1 \circ \gamma_2], [\Pi_2 \circ \gamma_2])$. Let $G = \Pi_1 \circ F$. G is continuous since it is a composition of continuous functions. We also have $G(s,0) = \Pi_1(F(s,0)) = \Pi_1 \circ \gamma_1$ and $G(s,1) = \Pi_1(F(s,1)) = \Pi_1 \circ \gamma_2$, so $\Pi_1 \circ \gamma_1 \sim_p \Pi_1 \circ \gamma_2$. Similarly, $\Pi_2 \circ \gamma_1 \sim_p \Pi_2 \circ \gamma_2$, so $\psi([\gamma_1]) = \psi([\gamma_2])$, hence ψ is well-defined.

ψ is a homomorphism since

$$
\begin{aligned}
\psi([\gamma_1][\gamma_2]) &= \psi([\gamma_1 * \gamma_2]) \\
&= (\Pi_{1*}([\gamma_1 * \gamma_2]), \Pi_{2*}([\gamma_1 * \gamma_2]) \\
&= ([\Pi_1 \circ (\gamma_1 * \gamma_2)]), [\Pi_2 \circ (\gamma_1 * \gamma_2)]) \\
&= ([(\Pi_1 \circ \gamma_1) * (\Pi_1 \circ \gamma_2)], [(\Pi_2 \circ \gamma_1) * (\Pi_2 \circ \gamma_2)]) \\
&= (\Pi_{1*}([\gamma_1])\Pi_{1*}([\gamma_2]), \Pi_{2*}([\gamma_1])\Pi_{2*}([\gamma_2])) \\
&= \psi([\gamma_1])\psi([\gamma_2])
\end{aligned}
$$

Consider any $(g, h) \in \pi_1(X, x) \times \pi_1(Y, y)$ and let $[\gamma_1] = g$ and $[\gamma_2] = h$ for some representatives γ_1 and γ_2. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. This is a well-defined loop in $X \times Y$ since $\gamma(0) = \gamma(1)$ and since γ is continuous as the coordinate functions are continuous. We have $\psi([\gamma]) = (\Pi_1([\gamma]), \Pi_2([\gamma])) = ([\Pi_1 \circ \gamma], [\Pi_2 \circ \gamma]) = ([\gamma_1], [\gamma_2]) = (g, h)$, hence ψ is surjective.

Consider any element g of the kernel of ψ and let $[\gamma] = g$ for some representative γ. Then $\psi([\gamma]) = ([\Pi_1 \circ \gamma], [\Pi_2 \circ \gamma]) = ([e_x], [e_y])$. Hence, $\gamma \sim_p e_{x \times y}$, since we can take the product of the path-homotopies $\Pi_1 \circ \gamma \sim_p e_x$ and $\Pi_2 \circ \gamma \sim_p e_y$ and run them on each corresponding coordinate. Thus ψ has trivial kernel, so it is injective. ∎

Proposition 88. *For continuous loops γ_1, γ_2 on a topological group G based at the identity element e, define $\gamma_1 \diamond \gamma_2$ by $\gamma_1 \diamond \gamma_2(x) = \gamma_1(x) \cdot \gamma_2(x)$, where the dot denotes the group operation. This loop is continuous and path-homotopic to $\gamma_1 * \gamma_2$. This induces a binary operation $\diamond$ on $\pi_1(G, e)$ which is the same as $*$.*

Proof. Since $\gamma_1 \diamond \gamma_2 = \cdot \circ (\gamma_1 \times \gamma_2)$ where $\cdot$ denotes the group operation which is continuous for a topological group, it is a composition of continuous functions, hence continuous. Additionally, $\gamma_1 \diamond \gamma_2(0) = \gamma_1 \diamond \gamma_2(1) = e$, so it is also a loop based at e.

Let $F(t_1, t_2) = \gamma_1(t_1) \cdot \gamma_2(t_2)$. For $t_1 = t_2$, we have $\gamma_1 \diamond \gamma_2$. For $t_2 = 0$ followed by $t_1 = 1$ we have $\gamma_1 * \gamma_2$. Deforming the diagonal into the bottom-edge followed by the right-edge shows they are path-homotopic.

Since $\diamond$ and $*$ behave the same on equivalence classes, they are the same operation on $\pi_1(G, e)$. ■

Proposition 89. *$\gamma_1 \diamond \gamma_2$ is also path-homotopic to $\gamma_2 * \gamma_1$.*

Proof. Using the same F as in the proof of **Proposition 88**, we see the path $t_1 = 0$ followed by $t_2 = 1$ gives $\gamma_2 * \gamma_1$. ■

Proposition 90. *For every topological group G, the fundamental group $\pi_1(G, e)$ is abelian.*

Proof. By **Propositions 88 & 89**, $[\gamma_1][\gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_1 \diamond \gamma_2] = [\gamma_2 * \gamma_1] = [\gamma_2][\gamma_1]$. ■

Proposition 91. *If X is path-connected and $h : X \to Y$ is a homeomorphism, then Y is path-connected.*

Proof. If Y has fewer than two points, then it is vacuously path-connected. Otherwise, consider $y_1, y_2 \in Y$. Since X is path-connected, there exists a path γ from $h^{-1}(y_1)$ to $h^{-1}(y_2)$. Then $h \circ \gamma$ is a path from y_1 to y_2 since it is continuous as the composition of two continuous functions and $(h \circ \gamma)(0) = h(h^{-1}(y_1)) = y_1$ and $(h \circ \gamma)(1) = h(h^{-1}(y_2)) = y_2$, where we have used that h is bijective. ■

Proposition 92. *If M is locally m-Euclidean, then for each $p \in M$, there exists an open neighborhood of p homeomorphic to $B_\delta(x) \subset \mathbb{R}^m$ for some $x \in \mathbb{R}^m, \delta \in \mathbb{R}^+$.*

Proof. For any $p \in M$, there exists an open neighborhood U of p homeomorphic to some open $V \subset \mathbb{R}^m$. Since open balls form a basis for the standard topology on $\mathbb{R}^m$, $V = \bigcup_{\lambda \in \Lambda} B_{\delta_\lambda}(x_\lambda)$ for some index set Λ. Let $B_{\delta_{\lambda_0}}(x_{\lambda_0})$ contain the image of p under the homeomorphism. Then the inverse image of $B_{\delta_{\lambda_0}}(x_{\lambda_0})$ under the homeomorphism contains p, is open by continuity of the homeomorphism, and is homeomorphic to $B_{\delta_{\lambda_0}}(x_{\lambda_0})$. ■

Proposition 93. *A locally m-Euclidean space M is path-connected if and only if it is connected.*

Proof. ($\implies$) Suppose for a contradiction there exist U, V, open, nonempty, and disjoint that cover M. Consider $x \in U, y \in V$, and γ a path from x to y. Then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are disjoint since U amd V are disjoint, and $\gamma^{-1}(U) \cup \gamma^{-1}(V) = \gamma^{-1}(U \cup V) = \gamma^{-1}(M) = [0, 1]$. Furthermore, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are open and nonempty, since γ is continuous and $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$. But then $[0, 1]$ is disconnected.

($\impliedby$) If M is empty, then it is vacuously path-connected. Otherwise, consider $x \in M$ and let P denote the set of points in M path-connected to x. Consider any point $y \in P$. M is locally Euclidean, so there exists an open neighborhood U of y homemorphic to some ball in $\mathbb{R}^n$ by **Proposition 92**. By **Proposition 91**, U must be path-connected, and in particular, $U \subset P$ since y is path-connected to x and to every point in U, so x is path-connected to every point in U. Hence P is open.

If the complement of P is empty, then $P = M$ so M is path-connected. Otherwise, consider $q \in M \backslash P$. As before, q contains a path-connected open neighborhood W

since M is locally Euclidean. If any point in W were path-connected to x, then q would be path-connected to x since W is path-connected, so we must have $W \subset M\backslash P$, so $M\backslash P$ is open. P nonempty, clopen in M connected implies $P = M$. ∎

Proposition 94. *If X is path-connected and $x_0 \in X$, then $\pi_1(X, x_0) \cong 1$ if and only if any two paths in X with the same endpoints are path-homotopic.*

Proof. If X has fewer than two points, the proposition follows immediately, so assume X contains at least two points.

($\Longrightarrow$) Let γ_1, γ_2 be paths from x_1 to x_2 in X. Then $\gamma_1 * \widehat{\gamma_2}$ is a loop based at x_1, so contractible since $\pi_1(X, x)$ is trivial for all basepoints x since it is trivial at x_0 and X is path-connected. Since $\gamma_1 * \widehat{\gamma_2} \sim_p e_{x_1}$, we have $\gamma_1 * \widehat{\gamma_2} * \gamma_2 \sim_p e_{x_1} * \gamma_2$ since path-homotopy is an equivalence relation and $*$ is well defined on the equivalence classes. But this gives $\gamma_1 \sim_p \gamma_2$.

($\Longleftarrow$) Let γ_1, γ_2 be two loops based at x_0. Since they are paths with the same endpoints, they must be path-homotopic, so $\pi_1(X, x_0) \cong 1$. ∎

Proposition 95. *For $n \in \mathbb{N}\backslash\{0\}$, the map $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ defined by $f(z) = z^n$ is a covering map.*

Proof. Let $U = \mathbb{C}\backslash\{x \in \mathbb{R} | x \geqslant 0\}$. Let V_k for $0 \leqslant k < n$ to be the set of all points in $\mathbb{C}\backslash 0$ that can be written as $re^{i\phi}$ with $r > 0$ and $\frac{2\pi k}{n} < \phi < \frac{2\pi(k+1)}{n}$. Let $g(re^{i\phi}) = \frac{1}{n} re^{i\frac{\phi}{k}}$. g is a continuous inverse of $f|_U$ therefore $f|_{V_k}$ is a homeomorphism between V_k and U. Hence, the V_k evenly cover U. Let $U_\theta = \mathbb{C}\backslash\{re^{i\theta} | r \geqslant 0\}$, then $\varphi_\theta(z) = ze^{i\theta}$ is a homeomorphism from U to U_θ. Let $V_{k,\theta} = \varphi_{\frac{\theta}{n}}(V_k)$, then $f|_{V_{\theta,k}} = \varphi^{-1}(f(\varphi_{\frac{\theta}{n}}(z)))$ is also a homeomorphism, so $f^{-1}(\mathbb{C}\backslash V_{\theta,k}) = U_\theta$, so $V_{\theta,k}$ evenly covers U_θ. ∎

Proposition 96. *If $p : Y \to X$ is a covering map, X is connected, and $p^{-1}(x)$ has k elements for one $x \in X$, then $p^{-1}(x)$ has k elements for all $x \in X$.*

Proof. Let $K \subset X$ be the set of k-covered elements. K is non-empty by assumption. Consider any element $x \in K$. Since p is a covering map, there exists an open neighborhood U of p such that $p^{-1}(U) \cong U \times \Lambda$ for some discrete set Λ. But since $p^{-1}(x)$ has k elements, we must have $\Lambda \cong \langle k \rangle$. Hence, every $y \in U$ is k-covered, so $U \subset K$, thus K is open. A similar argument shows $X\backslash K$ is open, so K is nonempty, clopen in X connected, hence $K = X$. ∎

Proposition 97. *If $f : Z \to Y$ and $g : Y \to X$ are covering maps such that for all $x \in X, g^{-1}(x)$ is finite, then $g \circ f$ is a covering map.*

Proof. If X is empty, the proposition follows vacuously. Otherwise, consider any $x \in X$. Since g is a covering map, there exists an open neighborhood U of x such that $g^{-1}(U) = \bigcup_{\lambda \in \Lambda} V_\lambda$, where the V_λ are disjoint and $g|_{V_\lambda} : V_\lambda \to U$ is a homeomorphism for all $\lambda \in \Lambda$.

Let $v_\lambda = g^{-1}(x) \cap V_\lambda$ for all $\lambda \in \Lambda$. Each v_λ has an evenly covered open neighborhood W_λ since f is a covering map. Let $K = \bigcap_{\lambda \in \Lambda} g(W_\lambda \cap V_\lambda)$. For all $\lambda \in \Lambda$, $g(W_\lambda \cap V_\lambda)$ is open since $W_\lambda \cap V_\lambda$ is an intersection of two open sets and g is a local homeomorphism, in particular open. Λ is finite since $g^{-1}(x)$ is finite, so K is open since it is a finite intersection of open sets $g(W_\lambda \cap V_\lambda)$. Since $v_\lambda \in W_\lambda \cap V_\lambda$ for all $\lambda \in \Lambda$, $x \in K$.

Let each $f^{-1}(W_\lambda) = \bigcup_{\delta\in\Delta} H_\lambda^\delta$ be the even cover of each W_λ. Then

$$(g \circ f)^{-1}(K) = \bigcup_{\lambda\in\Lambda,\delta\in\Delta} q^{-1}(W_\lambda \cap V_\lambda) \cap H_\lambda^\delta$$

This provides an even cover of K under $g \circ f$. ■

Proposition 98. *$S^1 \subset \mathbb{R}^2$ is a deformation retract of $\mathbb{R}^2\backslash\{0\}$.*

Proof. Let $H : \mathbb{R}^2\backslash\{0\} \times I \to \mathbb{R}^2\backslash\{0\}$ be given by $H(x, t) = (1-t)x + t\dfrac{x}{\|x\|}$. H is well-defined since for all $x \in \mathbb{R}^2\backslash\{0\}$, $\|x\| \neq 0$. It is known from analysis H is continuous. For all $v \in S^1$, $\|v\| = 1$, so $H(v, t) = v - tv + tv/1 = v$. Since $H(x, 0) = x$ and $H(v, 1) = v/\|v\|$, it is a deformation retraction.

■

Proposition 99. *$S^1 \subset \mathbb{R}^2$ cannot be a deformation retract of $\mathbb{R}^2$.*

Proof. By **Proposition 100**, if S^1 were a deformation retract of $\mathbb{R}^2$, its fundamental group would be trivial, but its fundamental group is $\mathbb{Z}$, which is not isomorphic to the trivial group. ■

Proposition 100. *If $f : X \to A$ is a deformation retract and $a \in A$, then the induced homomorphism $f_* : \pi_1(X, a) \to \pi_1(A, a)$ is an isomorphism.*

Proof. f_* is surjective by **Proposition 81**. Suppose $[\gamma]$ is in the kernel of f_*. If H is the given deformation retraction, then $H(\gamma, 0) = \gamma$ and $H(\gamma, 1) = f \circ \gamma$. Since the base-point is inside the retract, it remains fixed, so γ is path-homotopic to $f \circ \gamma$. But $f \circ \gamma$ is path-homotopic to the trivial loop since $[\gamma]$ is in the kernel of f_*, so by transitivity of path-homotopy, γ is path-homotopic to the trivial loop, hence the kernel of f_* is trivial, and thus f_* is injective. But a homomorphism that is a bijection is an isomorphism. ■

Proposition 101. *Let x_0 be any point in $\mathbb{R}^2\backslash\{0\}$. x_0 is a retract of $\mathbb{R}^2$ given by the constant map, but x_0 cannot be a deformation retract.*

Proof. If x_0 were a deformation retract of $\mathbb{R}^2\backslash\{0\}$, its fundamental group would be isomorphic to $\mathbb{Z}$ by **Proposition 100** and **Proposition 98**, but x_0 has trivial fundamental group and $\mathbb{Z}$ is not isomorphic to the trivial group. ■

Proposition 102. *$\mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z}$.*

Proof. If the two groups were isomorphic, then $\mathbb{Z} \times \mathbb{Z}$ would be cyclic, since the image of 1 under the isomorphism would generate $\mathbb{Z}$. But any $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ only generates $\{(kn, km) | k \in \mathbb{Z}\}$, which cannot be all of $\mathbb{Z} \times \mathbb{Z}$. ■

Proposition 103. *No two of $S^3, S^2 \times S^1$, and $S^1 \times S^1 \times S^1$ are homeomorphic.*

Proof. It suffices to show no two of $1, \mathbb{Z} \times 1$, and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ are isomorphic, since homeomorphic spaces have isomorphic fundamental groups. Since $\mathbb{Z} \times 1 \cong \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ are not trivial, it suffices to show that $\mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, but this follows by a proof entirely similar to the proof of **Proposition 102**. ∎

Proposition 104. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 2$.*

Proof. Let $e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. It was proven by Michael Thaddeus that $\pi_1(\mathbb{R}^n \backslash \{0\}, e_0) \cong 1$ for $n > 2$. But $\pi_1(\mathbb{R}^2 \backslash \{0\}, e_0) \cong \mathbb{Z}$ by **Propositions 98 & 100**. But if $\mathbb{R}^2 \backslash \{0\}$ is not homeomorphic to $\mathbb{R}^n \backslash \{0\}$ for $n > 2$, then $\mathbb{R}^2$ cannot be homeomorphic to $\mathbb{R}^n$ for $n > 2$. ∎

Proposition 105. *Let $X = Y = S^1$. Let $p : S^1 \to S^1$ be given by $p(z) = z^n$ for some $n \in \mathbb{N}$. Then the group of deck transformations D is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

Proof. Michael Thaddeus showed that p is a Galois covering, and for any Galois covering, we have that the group of deck transformations is isomorphic to $\pi_1(X)/p_*(\pi_1(Y))$. But $\pi_1(X) \cong \mathbb{Z}$, and $p_*(\pi_1(Y)) \cong n\mathbb{Z}$ since the image of the loop winding around S^1 once winds around n times under the map. ∎

Proposition 106. *Let $Z = S^1 \times \{a, b\}$ with $\{a, b\}$ discrete and let $\pi : Z \to S^1$ be given by $\pi(z, a) = z$ and let $\pi(z, b) = z^2$. Let $z_0 = (1, a), z_1 = (1, b), z_2 = (-1, b)$ so $\pi^{-1}(1) = \{z_0, z_1, z_2\}$. Let X be the figure eight and let Y be the quotient of disjoint union of two copies of Z under the equivalence relation $z_0 \sim z_2', z_1 \sim z_1', z_2 \sim z_0'$. Then the group of deck transformations $D_{Y \to X}$ is trivial, which is not equivalent to permuting the fibers, hence the covering is not Galois.*

Proof. Each copy of Z has a group of deck transformations of order two, since the copy of S^1 that corresponds to $\pi_1(S^1, b)$ can be rotated by π. However, the two copies of S^1 cannot be interchanged since this would destroy loops. Then deck transformations on Y correspond to deck transformations on the two copies of Z, so we check the cases where the deck transformations on the copies of Z are both trivial, one of them is nontrivial, and both of them are nontrivial. Since the transformations where at least one of them is nontrivial break loops through z_1, it must be that the group of deck transformations is trivial. ∎

Proposition 107. *Let $X \subset \mathbb{R}^3$ be the union of the spheres of radius 1 centered at $(0, 0, 1)$ and $(0, 0, -1)$, respectively.*

Proof. Let U and V be the intersections of X with open balls of radius $\frac{3}{2}$ centered at $(0, 0, 1)$ and $(0, 0, -1)$ respectively. Then $X = U \cup V$ and U and V are path-connected. Additionally, the spheres of radius 1 whose union is X are deformation retracts of U and V and their intersection has a point as a deformation retract.

Hence, by Seifert-Van Kampen, the fundamental group is the free product with amalgamation $1 *_1 1$, which is trivial. ■

Proposition 108. *Let w be a word of minimal length in its equivalence class in the free group $G * H$. Then w is a sequence of alternating letters in G and H with no identity elements.*

Proof. Suppose w is a word of minimal length. If w contains identity elements, these elements can be removed to produce an equivalent word with fewer letters, but w is minimal, hence it cannot contain identity elements. Similarly, if w contained two adjacent letters in the same group, they could be replaced by their product yielding an equivalent word which would be shorter. But w is minimal, so this cannot happen. ■

Proposition 109. *Let F be the following algorithm:*

F receives as its input a word w in the free product $G * H$. F begins by removing all identity elements from w. Then, it iterates through a word w left to right. Every time two letters belonging to the same group are found adjacent, it replaces them with their product. It terminates when no adjacent letters from the same group exist in the word.

Running F on a word w yields the word of minimum length in the equivalence class of w.

Proof. F must always terminate on any word since words have finite length and F either reduces the length of the word each iteration by replacing two letters with one letter or removing identity elements, or it does nothing, in which case it terminates. $F(w)$ satisfies the conditions of **Proposition 108** since F removes the identity elements from words and terminates only when words are an alternating sequence of elements in G and H.

Since F cannot add new letters, $|F(w)| \leqslant w$. If w has no identity elements and is a sequence of alternating letters in G and H, then F terminates without editing w, since no identity elements exist to be removed and no two adjacent elements can be multiplied. If w and w' are in the same equivalence class, then $F(w) = F(w')$ since if they differ by any identity elements, those are removed by F anyway. If they differ by element multiplication, F exhausts all possible element multiplication, and since group multiplication is associative, the result is the same. ■

Proposition 110. *Any word in $G * H$ is represented by a unique word of minimal length.*

Proof. Let w be any word in the free product. Let w' be any word of minimal length in the same equivalence class as w. Then $w' = F(w') = F(w)$, so $F(w)$ has minimal length. But we also see that for any word of minimal length, $w' = F(w)$, so the word of minimal length is unique. ■

Proposition 111. *If G and H are nontrivial, then the free product $G * H$ is not abelian.*

Proof. If g and h are nontrivial elements, then $gh \neq hg$, so the free product is not abelian. ■

Proposition 112. *The free product with amalgamation $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ is abelian, where the map denoted is the identity map.*

Proof. The group is isomorphic to $\mathbb{Z}$ itself, so it is abelian, as we've seen in class. ■

Proposition 113. *The fundamental group of a bouquet of n circles is $\mathbb{Z}^{*n}$.*

Proof. We proceed by induction and Seifert-Van Kampen. For $n = 1$, the bouqet is homeomorphic to S^1, so it's fundamental group is $\mathbb{Z}$. For $n = 2$, it's homeomorphic to the figure-eight, so its fundamental group is $\mathbb{Z} * \mathbb{Z}$. Assume the proposition holds for $n = k$. For $n = k + 1$, we can decompose this as a union of a bouquet of k circles union another circle with their intersection being a single point. Of course to apply Seifert-Van Kampen we require open sets, but it's clear that our sets are deformation retracts of slightly larger open sets. The fundamental group is thus $\mathbb{Z}^{*k} *_1 \mathbb{Z} = \mathbb{Z}^{*(k+1)}$, so the proposition holds. ■

Proposition 114. *The fundamental group of the complement of n points in $\mathbb{R}^2$ is the same as the fundamental group of a bouquet of n circles.*

Proof. The complement of n points in $\mathbb{R}^2$ has a bouquet of n circles as a deformation retract, so the proposition follows. ■

Proposition 115. *The fundamental group of the union of a torus and a disk cutting through the torus is $\mathbb{Z}$.*

Proof. The disk has trivial fundamental group, while the torus has $\mathbb{Z} \times \mathbb{Z}$ as its fundamental group. Their intersection, S^1, has $\mathbb{Z}$ as its fundamental group. The torus and disk are deformation retracts of slightly larger open sets which are path connected and cover the space, as well as having the circle as a deformation retract of their intersection, so we have that the fundamental group of the space is $1 *_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}$ by Seifert-Van Kampen, which is isomorphic to $\mathbb{Z}$ since the homeomorphism is just an embedding of $\mathbb{Z}$ in $\mathbb{Z} \times \mathbb{Z}$. ■

Proposition 116. *The fundamental group of the complement of n points on a torus is the same as the fundamental group of a bouquet of $n + 1$ petals.*

Proof. This follows since the complement of n points on a torus has a bouquet of $n + 1$ petals as a deformation retract. ■